HOW DID RAMANUJAN CREATE A SERIES FOR PI?

ALEX THOLEN

1. Preliminary Definitions

Let us start by defining a lot of functions, which will come in use later on.

Definition 1.1. Define the rising factorial$(a)_n := a(a+1)(a+2)(a+3)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)}$.

Definition 1.2. Define ${}_2F_1(a,b;c;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!}$ and ${}_3F_2(a,b,c;d,e;x) := \sum_{n=0}^{\infty} \frac{(a)_n(b)_n(c)_n}{(d)_n(e)_n}\frac{x^n}{n!}$.

Definition 1.3. Define $K_s(k) := \frac{\pi}{2}\cdot {}_2F_1(\frac{1}{2}-s,\frac{1}{2}+s;1;k^2)$ and $E_s(k) := \frac{\pi}{2}\cdot {}_2F_1(-\frac{1}{2}-s,\frac{1}{2}+s;1;k^2)$ where $|s| < \frac{1}{2}$ and $0 \leq k < 1$.

Definition 1.4. $k' := \sqrt{1-k^2}$, and $K'_s(k) := K_s(k')$, and $E'_s(k) := E_s(k')$. $K := K_0, E := E_0$.

Definition 1.5. Define $G := (2kk')^{-\frac{1}{12}}, g := (\frac{2k}{k'^2})^{-\frac{1}{12}}$.

Definition 1.6. Define $k(N) := k$ s.t. $\frac{K'}{K}(k(N)) = \sqrt{N}$.

Definition 1.7. Define $\alpha(N) := (\frac{E'}{K} - \frac{\pi}{4K^2})(k(N))$.

2. Necessary Theorems

Theorem 2.1. *? (invariants)*

Theorem 2.2. $E_s = k'^2 K_s + \frac{kk'^2}{1+2s}\dot{K}_s$.

Proof. Let's turn these all into sums. We need

$$\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(-\frac{1}{2}-s)_n(\frac{1}{2}+s)_n}{n!}\frac{k^{2n}}{n!}$$

$$= (1-k^2)\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2}-s)_n(\frac{1}{2}+s)_n}{n!}\frac{k^{2n}}{n!} + \frac{k-k^3}{1+2s}\frac{\pi}{2}\sum_{n=0}^{\infty}\frac{(\frac{1}{2}-s)_n(\frac{1}{2}+s)_n}{n!}\frac{2nk^{2n-1}}{n!}$$

This is true - proof by desmos https://www.desmos.com/calculator/8qxzgmy9xp (I know this doesn't work, I just can't prove it). ■

Theorem 2.3. $E_sK'_s + K_sE'_s - K_sK'_s = \frac{\pi}{2}\frac{\cos(\pi s)}{1+2s}$.

Proof. This is proven in Erdeyli ■

Theorem 2.4. $\frac{2}{\pi}K_s(h) = {}_2F_1(\frac{1}{4}-\frac{s}{2},\frac{1}{4}+\frac{s}{2};1;(2hh')^2)$
$(\frac{2}{\pi}K_s(h))^2 = {}_3F_2(\frac{1}{2}-s,\frac{1}{2}+s,\frac{1}{2};1,1;(2hh')^2)$

Proof. ? ■

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Theorem 2.5.

$$\frac{2K}{\pi}(k) = {}_2F_1(\frac{1}{4},\frac{1}{4};1;(2kk')^2), \quad (0 \le k \le \frac{1}{\sqrt{2}}) \tag{2.1}$$

$$= k^{-\frac{1}{2}}\ {}_2F_1(\frac{1}{4},\frac{1}{4};1;-(2k/k'^2)^2), \quad (0 \le k \le \sqrt{2}-1) \tag{2.2}$$

$$= k^{-\frac{1}{2}}\ {}_2F_1(\frac{1}{4},\frac{1}{4};1;-(\frac{k^2}{2k'})^2), \quad (0 \le k^2 \le 2\sqrt{2}-2) \tag{2.3}$$

$$\frac{2K}{\pi}(k) = (1+k^2)^{-\frac{1}{2}}\ {}_2F_1(\frac{1}{8},\frac{3}{8};1;(\frac{g^{12}+g^{-12}}{2})^{-2}) \quad (0 \le k \le \sqrt{2}-1) \tag{2.4}$$

$$= (k'^2-k^2)^{-\frac{1}{2}}\ {}_2F_1(\frac{1}{8},\frac{3}{8};1;-(\frac{G^{12}-G^{-12}}{2})^{-2}) \quad (0 \le k \le \frac{2^{\frac{1}{4}}-\sqrt{2-\sqrt{2}}}{2}) \tag{2.5}$$

$$\frac{2K}{\pi}(k) = (1-(kk')^2)^{-\frac{1}{4}}\ {}_2F_1(\frac{1}{12},\frac{5}{12};1;J^{-1}) \quad (0 \le k \le \frac{1}{\sqrt{2}}) \tag{2.6}$$

Proof. The first one is just plugging in $s = 0$. ∎

Theorem 2.6. *Under the same restrictions as 2.5 we have*

$$(\frac{2K}{\pi}(k))^2 = {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;(2kk')^2) \tag{2.7}$$

$$= k'^{-2}\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-(\frac{2k}{k'^2})^2) \tag{2.8}$$

$$= k'^{-1}\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,1;-(\frac{k^2}{2k'})^2) \tag{2.9}$$

$$(\frac{2K}{\pi}(k))^2 = (1+k^2)^{-1}\ {}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;(\frac{g^{12}+g^{-12}}{2})^{-2}) \tag{2.10}$$

$$= (k'^2-k^2)^{-1}\ {}_3F_2(\frac{1}{4},\frac{3}{4},\frac{1}{2};1,1;-(\frac{G^{12}-G^{-12}}{2})^{-2}) \tag{2.11}$$

$$(\frac{2K}{\pi}(k))^2 = (1-(kk')^2)^{-\frac{1}{2}}\ {}_3F_2(\frac{1}{6},\frac{5}{6},\frac{1}{2};1,1;J^{-1}) \tag{2.12}$$

Proof. 2.3+2.4 ∎

Theorem 2.7. *One recursion for α is $\alpha(4N) = \frac{4\alpha(N)-2\sqrt{N}k^2(N)}{(1+k'(N))^2}$.*

Proof. ? ∎

Theorem 2.8. *A formula for $\frac{1}{\pi}$ is the following: $\frac{1}{\pi} = \sqrt{N}kk'^2\frac{4K\dot{K}}{\pi^2} + (\alpha(N) - \sqrt{N}k^2)\frac{4K^2}{\pi^2}(k := k(N))$*

Proof. 2.3 ∎

Theorem 2.9. $\frac{4K\dot{K}}{\pi^2} = \frac{1}{2}\dot{m}F + \frac{1}{2}m\dot{\phi}\dot{F}(\phi)$ *for some algebraic m, ϕ and where $F = \sum_{n=0}^{\infty} a_n\phi^n$.*

3. Final Stretch

Theorem 3.1.

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} a_n \left(\frac{\sqrt{N}}{2}kk'^2\dot{m} + (\alpha(N) - \sqrt{N}k^2)m + n\frac{\sqrt{N}}{2}m\frac{\dot{\phi}}{\phi}kk'^2 a \right) \phi^n$$

where m, ϕ are algebraic functions and $\dot{x} = \frac{dx}{dt}$.

Proof. We begin with 2.9. Expanding the F's gets

$$\frac{4K\dot{K}}{\pi} = \sum_{n=0}^{\infty} \frac{1}{2}\dot{m}a_n\phi^n + \sum_{n=0}^{\infty} \frac{1}{2}m\dot{\phi}a_{n+1}(n+1)\phi^n.$$

Combining the sums gets $\frac{1}{2}\dot{m}a_0 + \sum_{n=0}^{\infty} \frac{1}{2}a_n\phi^n(\phi\dot{m} + (n+1)\dot{\phi})$ 2.8 ■

Kinda General Form:
$x_N := \frac{2}{g_N^{12}+g_N^{-12}} = \frac{4k(N)k'^2(N)}{(1+k^2(N))^2}$
$d_n(N) := \left[\frac{\alpha(N)x_N^{-1}}{1+k^2(N)} - \frac{\sqrt{N}}{4}g_N^{-12}\right] + n\sqrt{N}\left[\frac{g_N^{12}-g_N^{-12}}{2}\right]$

$$\frac{1}{\pi} = \sum_{n=0} \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3} d_n(N)x_N^{2n+1}$$

Proof. Let's begin with the formula for $\frac{1}{\pi}$ used in 3.1. We have

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} a_n \left(\frac{\sqrt{N}}{2}kk'^2\dot{m} + (\alpha(N) - \sqrt{N}k^2)m + n\frac{\sqrt{N}}{2}m\frac{\dot{\phi}}{\phi}kk'^2 a\right)\phi^n$$

$+$ 2.1 ■

Theorem 3.2. *Given* $g_N^{-12} = (\frac{\sqrt{29}-5}{2})^6$ *and* $\alpha(N) = (\frac{\sqrt{29}+5}{2})^6(99\sqrt{29} - 444)(99\sqrt{2} - 70 - 13\sqrt{29})$. *These are* $N = 58$. *we get*

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(\sqrt{8}(1103 + 26390n))\left(\frac{1}{99^2}\right)^{2n+1}$$

Proof. Let's expand g_N^{-12}. We get $g_N^{-12} = 9801 - 1820\sqrt{29}$. Expanding $\alpha(N)$ gets $\alpha(N). = -68403 - 948109536\sqrt{2} + 12702\sqrt{29} + 176059521\sqrt{58}$ Yikes. First off, the factorials don't change. So what needs to be shown is that for $N = 58$ we get that $d_n(N)x_N^{2n+1} = (\sqrt{8}(1103+26390n))(\frac{1}{99^2})^{2n+1}$. We defined $x_N = \frac{2}{g_N^{12}+g_N^{-12}}$, and it turns out that after simplifying we get $x_N = \frac{1}{99^2}$. That means we need that $d_n(58) = \sqrt{8}(1103 + 26390n)$. Let's first look at the second part. We can simplify $\frac{g_N^{12}-g_N^{-12}}{2}$ into $1820\sqrt{29}$. Multiplying this by $n\sqrt{58}$ gets $52780n\sqrt{2} = 26390n\sqrt{8}$. So what that leaves is $1103\sqrt{8} = \frac{\alpha(N)x_N^{-1}}{1+k^2} - \frac{\sqrt{58}}{4}g_N^{-12}$. We know most of these, but we don't know k. Well, $\frac{2k(N)}{k'(N)^2} = g_N^{-12}$, and so $\frac{2k(N)}{1-k(N)^2} = (\frac{\sqrt{29}-5}{2})^6$. That gets either $k(N) = \frac{g_N^{-12}}{1+99\sqrt{2g_N^{-12}}}$ or $k(N) = (-9801 - 1820\sqrt{29})(1 + 99\sqrt{2g_N^{-12}})$. Since $k(N) < 1$, it is the first one. So we need $1103\sqrt{8} = \frac{(\frac{\sqrt{29}+5}{2})^6(99\sqrt{29}-444)(99\sqrt{2}-70-13\sqrt{29})\cdot 9801}{(1+(\frac{9801-1820\sqrt{29}}{1+99\sqrt{19602-3640\sqrt{29}}})^2)} - \frac{\sqrt{58}}{4}(\frac{\sqrt{29}-5}{2})^6$. After many many stages of simplifying, it turns out that this abomination of an equality is true. ■

From what is shown above we can make a few substitutions to bring it into the form we most recognize. The easy ones are to bring out the $\frac{2\sqrt{2}}{99^2}$ - that makes it into

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2}\sum_{n=0}^{\infty} \frac{(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n}{(n!)^3}(1103 + 26390n)\left(\frac{1}{99^2}\right)^{2n}.$$

After this we deal with $(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n$. Multiplying these each by 4^n gets $1(1+4)(1+8)(1+12)\cdots(1+4(n-1))2(2+4)(2+8)(2+12)\cdots(2+4(n-1))3(3+4)(3+8)(3+12)\cdots(3+4(n-1))$. All that is missing now is $4^n n!$, which makes the whole product become $4n!$. So, we have that $(\frac{1}{4})_n(\frac{1}{2})_n(\frac{3}{4})_n 4^{4n} n! = 4n!$, and so we now have the equation

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2}\sum_{n=0}^{\infty}\frac{4n!}{4^{4n}n!^4}\frac{1103+26390n}{99^{4n}}.$$

We can combine the 4^{4n} with the 99^{4n} and that becomes the equation that is well known -

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{99^2}\sum_{n=0}^{\infty}\frac{(4n)!}{n!^4}\frac{26390n+1103}{396^{4n}}$$

Primary Source: http://www.cecm.sfu.ca/personal/pborwein/PAPERS/P36.pdf
References: Erdeyli, A. et al., *Higher Transcendental Function*s, McGraw-Hill, 1953.