ARITHMETIC-GEOMETRIC MEAN

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1. Summary/Background

Most people are very comfortable with the geometric and arithmetic means due to their secondary education, but little have explored the concept of the arithmetic-geometric mean, and how it can be used to explore surprisingly related topics in math. The arithmetic-geometric mean first appeared in famous mathematicians such as Lagrange and Gauss' papers during the late 18th century.

Throughout this paper, we will define and prove the existence of the arithmetic-geometric mean as well as prove a few statements involving them. You may be surprised to see what relation the arithmetic-geometric mean has with elliptic integrals and yet another approximation for π.

2. Definition and Existence

Definition 2.1. Take two positive real numbers (not necessarily distinct) a_0 and b_0. Define the recurrence relation such that for all $n \geq 0$ and $n \in \mathbb{N}$,

$$a_{n+1} = \frac{a_n + b_n}{2}$$

and

$$b_{n+1} = \sqrt{a_n b_n}.$$

The *arithmetic-geometric mean* of the two numbers a_0 and b_0 is equal to

$$\lim_{n\to\infty} g_n = \lim_{n\to\infty} a_n.$$

Proof of Existence. For any two positive real numbers a and b, it is known that by the Arithmetic Mean - Geometric Mean (AM-GM) Inequality, $\frac{a+b}{2} \geq \sqrt{ab}$. Thus, as defined above, $b_n \geq a_n$ for all nonnegative integers n. This in turn implies that

$$b_{n+1} = \sqrt{b_n a_n} \geq \sqrt{b_n b_n} = b_n.$$

Since the sequence b_i is monotonically increasing, but is bounded by $\max{(a_0, b_0)}$, by the monotone convergence theorem, the sequence must converge. This also allows for there to exist a constant g such that

$$\lim_{n\to\infty} b_n = g.$$

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Now that the convergence of $\{b_i\}$ is known, this can be used to compute $\lim_{n\to\infty} a_n$ as such:

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{b_{n+1}^2}{b_n} = \frac{\lim_{n\to\infty} b_{n+1}^2}{\lim_{n\to\infty} b_n} = \frac{g^2}{g} = g.$$

Thus, since both $\{a_i\}$ and $\{b_i\}$ converge to the same limit, namely the arithmetic-geometric mean, its existence has been proven. The arithmetic-geometric mean of x and y from now on will be denoted as $M(x, y)$.

□

3. Properties/Theorems

Theorem 3.1. For all positive constants c, we have $(cx, cy) = cM(x, y)$.

Proof. The following statement is pretty obvious (assume $d \geq 0$; it follows simply from manually computing the arithmetic and geometric means of a_n and b_n):

$$a_n = dx, b_n = dy \implies a_{n+1} = d\left(\frac{x+y}{2}\right), b_{n+1} = d\sqrt{xy}.$$

This implies the following:

$$\text{AM}(dx, dy) = d\text{AM}(x, y)$$

and

$$\text{GM}(dx, dy) = d\text{GM}(x, y).$$

Since the arithmetic-geometric mean (AGM) is simply repeated iterations of the arithmetic and geometric means, and both functions satisfy $d \cdot f(x, y) = f(dx, dy)$), the AGM must also satisfy this property, thus completing our proof.

□

Definition 3.2. The *complete elliptic integral of the first kind* is defined as

$$K(k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}}.$$

where $0 \leq k \leq 1$. If $k = 1$, we get ∞.

Theorem 3.3. For the Arithmetic-Geometric Mean function $M(x, y)$, we have the following identity:

$$M(x, y) = \frac{\pi}{2} \div \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{x^2 \cos^2\theta + y^2 \sin^2\theta}}.$$

Proof. The following proof was utilized by Gauss and is due to [Cox97].

Let us define the function I to be the following:

$$I\left(x,y\right)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{x^2\cos^2\theta+y^2\sin^2\theta}}.$$

Additionally, let $x<y$ be constants and let x_1 and y_1 be the arithmetic and geometric means of x and y, respectively. Finally, define another variable θ' such that

$$\sin\theta=\frac{2x\sin\theta'}{\left(x+y\right)+\left(x-y\right)\sin^2\theta'}.$$

For sake of simplicity, set m equal to

$$\left(x+y\right)+\left(x-y\right)\sin^2\theta'.$$

Before directly approaching the theorem, we will first show two proofs of equivalences that are necessary to prove the theorem, namely

$$\cos\theta=\frac{2\cos\theta'\sqrt{x_1^2\cos^2\theta'+y_1^2\sin^2\theta'}}{m}$$

and

$$\sqrt{x^2\cos^2\theta+y^2\sin^2\theta}=x\cdot\frac{x+y-\left(x-y\right)\sin^2\theta'}{m}.$$

To prove the first one, we square the equation we have for $\sin\theta$.

$$\sin\theta=\frac{2x\sin\theta'}{m}.$$

$$\sin^2\theta=\frac{4x^2\sin^2\theta'}{m^2}.$$

$$\cos^2\theta=1-\sin^2\theta=1-\frac{4x^2\sin^2\theta'}{m^2}=$$

$$\frac{x^2+2xy+y^2+2\left(x+y\right)\left(x-y\right)\sin^2\theta'+x^2\sin^4\theta'-2xy\sin^4\theta'+y^2\sin^4\theta'-4x^2\sin^2\theta'}{m^2}=$$

$$\frac{x^2+2xy+y^2+2\left(x^2-y^2-2x^2\right)\sin^2\theta'+x^2\sin^4\theta'-2xy\sin^4\theta'+y^2\sin^4\theta'}{m^2}=$$

$$\frac{x^2+2xy+y^2-2\left(x^2+y^2\right)\sin^2\theta'+x^2\sin^4\theta'-2xy\sin^4\theta'+y^2\sin^4\theta'}{m^2}=$$

$$\frac{x^2\left(1-2\sin^2\theta'+\sin^4\theta\right)+2xy\left(1-\sin^4\theta'\right)+y^2\left(1-2\sin^2\theta'+\sin^4\theta'\right)}{m^2}=$$

$$\frac{x^2\left(1-\sin^2\theta'\right)^2+y^2\left(1-\sin^2\theta'\right)^2+2xy\left(1+\sin^2\theta'\right)\left(1-\sin^2\theta'\right)}{m^2}=$$

$$\frac{x^2\left(\cos^2\theta'\right)^2+y^2\left(\cos^2\theta'\right)^2+2xy\left(1+\sin^2\theta'\right)\left(\cos^2\theta'\right)}{m^2}=$$

$$\frac{\cos^2\theta'}{m^2}\cdot\left(x^2\cos^2\theta'+y^2\cos^2\theta'+2xy\left(1+\sin^2\theta'\right)\right)=$$
$$\frac{\cos^2\theta'}{m^2}\cdot\left(x^2\cos^2\theta'+y^2\cos^2\theta'+2xy\cos^2\theta'+4xy\sin^2\theta'\right)=$$
$$\frac{4\cos^2\theta'}{m^2}\cdot\left(\frac{x^2\cos^2\theta'+2xy\cos^2\theta'+y^2\cos^2\theta'}{4}+xy\sin^2\theta'\right)=$$
$$\frac{4\cos^2\theta'}{m^2}\cdot\left(\left(\frac{x+y}{2}\right)^2\cos^2\theta'+(\sqrt{xy})^2\sin^2\theta'\right).$$

Now we take the square root to get our desired equality:

$$\cos\theta=\sqrt{\frac{4\cos^2\theta'}{m^2}\cdot\left(\left(\frac{x+y}{2}\right)^2\cos^2\theta'+(\sqrt{xy})^2\sin^2\theta'\right)}=$$
$$\frac{2\cos\theta'\sqrt{\left(\frac{x+y}{2}\right)^2\cos^2\theta'+\left(\sqrt{xy}\right)^2\sin^2\theta'}}{m}=\frac{2\cos\theta'\sqrt{x_1^2\cos^2\theta'+y_1^2\sin^2\theta'}}{m}.$$

For the second equality, we just substitute are values for $\sin^2\theta$ and $\cos^2\theta$.

$$\sin^2\theta=\frac{4x^2\sin^2\theta'}{m^2}.$$

$$\cos^2\theta=\frac{x^2+2xy+y^2-2\left(x^2+y^2\right)\sin^2\theta'+x^2\sin^4\theta'-2xy\sin^4\theta'+y^2\sin^4\theta'}{m^2}.$$

$$\sqrt{x^2\cos^2\theta+y^2\sin^2\theta}=$$

$$\sqrt{\frac{x^4+2x^3y+x^2y^2-2\left(x^4+x^2y^2\right)\sin^2\theta'+x^4\sin^4\theta'-2x^3y\sin^4\theta'+x^2y^2\sin^4\theta'}{m^2}+\frac{4x^2y^2\sin^2\theta'}{m^2}}=$$

$$\sqrt{\frac{x^4+2x^3y+x^2y^2-2\left(x^4-x^2y^2\right)\sin^2\theta'+x^4\sin^4\theta'-2x^3y\sin^4\theta'+x^2y^2\sin^4\theta'}{m^2}}=$$

$$\frac{x}{m}\sqrt{x^2+2xy+y^2-2\left(x^2-y^2\right)\sin^2\theta'+x^2\sin^4\theta'-2xy\sin^4\theta'+y^2\sin^4\theta'}=$$

$$\frac{x}{m}\sqrt{(x+y)^2-2\left(x+y\right)\left(x-y\right)\sin^2\theta'+(x-y)^2\left(\sin^2\theta'\right)^2}=$$

$$\frac{x}{m}\sqrt{\left((x+y)-(x-y)\sin^2\theta'\right)^2}=$$

$$x\cdot\frac{x+y-(x-y)\sin^2\theta'}{m}.$$

Now that we have our two equalities, we differentiate both sides of our equation for $\sin\theta$.

$$\frac{d}{d\theta}(\sin\theta) = \frac{d}{d\theta'}\left(\frac{2x\sin\theta'}{(x+y)+(x-y)\sin^2\theta'}\right).$$

We can simply use a calculator for this; since x and y are treated as constants, our result is

$$\cos\theta = \frac{2x\left(\cos\theta'\left(x+y+\sin^2\theta'(x-y)\right) - 2\sin^2\theta'\cos\theta'(x-y)\right)}{m^2}.$$

We can equate this result with our expression for $\cos\theta$:

$$\frac{2x\left(\cos\theta'\left(x+y+\sin^2\theta'(x-y)\right) - 2\sin^2\theta'\cos\theta'(x-y)\right)}{m^2} = \frac{2\cos\theta'\sqrt{x_1^2\cos^2\theta' + y_1^2\sin^2\theta'}}{m}.$$

Dividing both sides by $\frac{2\cos\theta'}{m}$ gives us

$$\frac{x\left(x+y+\sin^2\theta'(x-y) - 2\sin^2\theta'(x-y)\right)}{m} = \sqrt{x_1^2\cos^2\theta' + y_1^2\sin^2\theta'}.$$

Simplifying a little more, we have

$$\frac{x\left(x+y-\sin^2\theta'(x-y)\right)}{m} = \sqrt{x_1^2\cos^2\theta' + y_1^2\sin^2\theta'}.$$

The left of the side of the equation (due to our second proven equation) is

$$x\cdot\frac{x+y-(x-y)\sin^2\theta'}{m} = \sqrt{x^2\cos^2\theta' + y^2\sin^2\theta'}.$$

Therefore, we conclude that

$$\sqrt{x^2\cos^2\theta' + y^2\sin^2\theta'} = \sqrt{x_1^2\cos^2\theta' + y_1^2\sin^2\theta'}.$$

Now we can use this equivalence to show that

$$I(x,y) = I(x_1,y_1) = I(x_2,y_2) = \ldots$$

If we continue with this iteration until its limit, we have

$$I(x,y) = \lim_{n\to\infty} I(x_n,y_n) = I\left(M(x,y), M(x,y)\right).$$

$$I\left(M(x,y), M(x,y)\right) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{M(x,y)^2\cos^2\theta + M(x,y)^2\sin^2\theta}} =$$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{M(x,y)} = \frac{\pi}{2M(x,y)}.$$

Thus,

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{x^2\cos^2\theta + y^2\sin^2\theta}} = \frac{\pi}{2M(x,y)}.$$

Some manipulation of both sides gives us our desired result

$$M\left(x,y\right)=\frac{\pi}{2}\div\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{x^2\cos^2\theta+y^2\sin^2\theta}}.$$

□

Proposition 3.4. For the Arithmetic-Geometric Mean function $M(x,y)$ and the complete elliptic integral of the first kind function $K(k)$ we have the following identity:

$$\frac{a\pi}{2M\left(a,b\right)}=K\left(\frac{1}{a}\sqrt{a^2-b^2}\right).$$

Proof. The following proof in part is due to [Gil].

We start out with the following equality:

$$\frac{\pi}{2M\left(a,b\right)}=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}.$$

We can manipulate the right hand side to get the following.

$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}=\int_0^{\frac{\pi}{2}}\frac{d\theta}{a\sqrt{\cos^2\theta+\frac{b^2}{a^2}\sin^2\theta}}=$$
$$\int_0^{\frac{\pi}{2}}\frac{d\theta}{a\sqrt{1-\frac{a^2-b^2}{a^2}\sin^2\theta}}=\frac{1}{a}\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-\frac{a^2-b^2}{a^2}\sin^2\theta}}=$$
$$\frac{1}{a}K\left(\frac{1}{a}\sqrt{a^2-b^2}\right).$$

Returning to our original equation, we have

$$\frac{\pi}{2M\left(a,b\right)}=\frac{1}{a}K\left(\frac{1}{a}\sqrt{a^2-b^2}\right).$$
$$\frac{a\pi}{2M\left(a,b\right)}=K\left(\frac{1}{a}\sqrt{a^2-b^2}\right).$$

□

Now that we have a working result, we can try out a few examples.

1. If $a=b\neq 0$ we have

$$\frac{a\pi}{2M\left(a,a\right)}=K\left(0\right).$$
$$\frac{a\pi}{2a}=K\left(0\right).$$
$$K\left(0\right)=\frac{\pi}{2}.$$

2. We can try finding the arithmetic-geometric mean of 1 and 0:

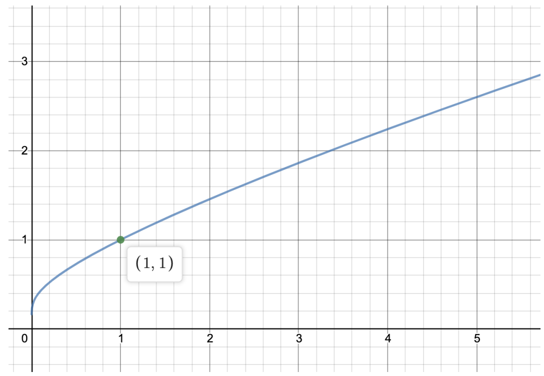

Figure 1. AGM of 1 and x graphed on Cartesian Plane

$$a_0 = 1, b_0 = 0.$$

$$a_1 = \frac{1}{2}, b_1 = 0.$$

$$a_2 = \frac{1}{4}, b_2 = 0.$$

We notice that the arithmetic-geometric mean will be equal to 0 since the geometric mean(b_is) is always 0. If we plug in $a = 1$ and $b = 0$, we get

$$\frac{\pi}{2M(1,0)} = K\left(\frac{1}{1}\sqrt{1^2 - 0^2}\right).$$

$$\frac{\pi}{0} = K(1).$$

Both sides tend toward ∞, as expected.

3. We can also graph the arithmetic-geometric mean of a and b. We use the direct result of Theorem 3.3,

$$M(x,y) = \frac{\pi}{2} \div \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{x^2\cos^2\theta + y^2\sin^2\theta}}.$$

Figure 1 shows the graph of the the arithmetic-geometric mean of 1 and x. Notice that the graph is between the two equations $y = \frac{x+1}{2}$ (the arithmetic mean) and $y = \sqrt{x}$ (the geometric mean). The first number in our AGM need not be 1; we can easily adjust this number using Theorem 3.1 while dividing the second number accordingly. For instance, to find the AGM of 3 and 4, we can compute the following:

$$3M\left(1, \frac{4}{3}\right) = M(3,4) \approx 3.482.$$

4. In fact, we can extend our graph from (3) to the 3D space, so that we can set $z = M(x,y)$. **Figure 2** shows this result.

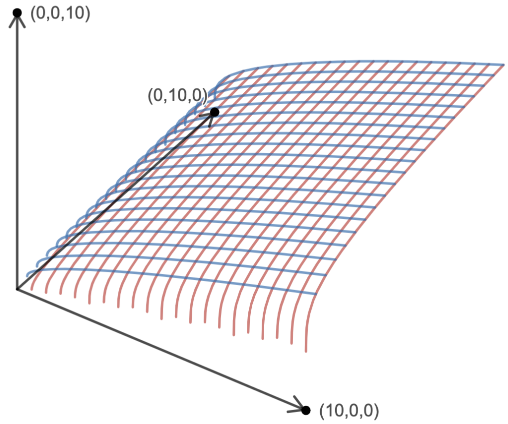

Figure 2. AGM of x and y in 3D Space

We now proceed to our final interesting result using the arithmetic geometric mean, which is an approximation for π.

Theorem 3.5.

$$\pi = \frac{4\left(M\left(1, \frac{1}{\sqrt{2}}\right)\right)^2}{1 - 2\sum_{j=1}^{\infty} 2^j c_j^2}$$

where $c_n^2 = a_n^2 - b_n^2$.

Before we proceed with the proof of Theorem 3.5, let us prove some intermediate results. All following propositions and proofs are due to [Mil19].

Proposition 3.6. Denote a function L such that

$$L(a, b) = \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} d\theta.$$

Then the function L satisfies the following properties:

(1) $L(b, a) + L(a, b) = I(a, b)$

(2) $L(b, a) - L(a, b) = \frac{a-b}{a+b} \cdot L(b_1, a_1)$

where the function I, as mentioned previously, is the following:

$$I(x, y) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{\sqrt{x^2\cos^2\theta + y^2\sin^2\theta}}.$$

Proof.

(1) can be proved by substituting $\theta' = \frac{\pi}{2} - \theta$. Then we have $\cos\theta' = \sin\theta$ and $\sin\theta' = \cos\theta$, which yields the following equations:

$$L(b,a) = \int_0^{\frac{\pi}{2}} \frac{\cos^2\theta}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} d\theta = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta'}{\sqrt{b^2\sin^2\theta' + a^2\cos^2\theta'}} d\theta'.$$

Since $\sin^2 + \cos^2 = 1$, we have

$$L(b,a) + L(a,b) = \int_0^{\frac{\pi}{2}} \frac{\sin^2\theta + \cos^2\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}} d\theta = I(a,b).$$

For the second equation, we start by substituting $t = b\tan\theta$. Our substitution is then $\theta = \arctan\left(\frac{t}{b}\right) \to d\theta = \frac{1}{b} \cdot \frac{1}{\left(\frac{t}{b}\right)^2 + 1} dt = \frac{b}{t^2 + b^2} dt$. Additionally, we have $\frac{b^2}{b^2+t^2} = \frac{1}{1+\sec^2\theta} = \cos^2\theta$ and $\frac{t^2}{b^2+t^2} = 1 - \cos^2\theta = \sin^2\theta$. Putting all of it together, we have the following:

$$L(a,b) = \int_0^{\infty} \frac{\frac{b^2}{b^2+t^2}}{\sqrt{a^2 \cdot \frac{b^2}{b^2+t^2} + b^2 \cdot \frac{t^2}{b^2+t^2}}} \cdot \frac{b}{t^2+b^2} dt = \int_0^{\infty} \frac{\frac{b^2}{b^2+t^2} dt}{\sqrt{(t^2+a^2)(t^2+b^2)}}.$$

We can do the same for calculating $L(b,a)$, simply replacing all "b"s with "a"s and vice versa. Therefore, we have

$$L(b,a) - L(a,b) = \int_0^{\infty} \frac{\frac{a^2}{b^2+t^2} - \frac{b^2}{b^2+t^2}}{\sqrt{(t^2+a^2)(t^2+b^2)}} dt.$$

With some simplification we have

$$\frac{a^2}{b^2+t^2} - \frac{b^2}{b^2+t^2} = \frac{a^2(b^2+t^2) - b^2(a^2+t^2)}{(a^2+t^2)(b^2+t^2)} = \frac{a^2t^2 - b^2t^2}{(a^2+t^2)(b^2+t^2)} = \frac{(a^2-b^2)t^2}{(a^2+t^2)(b^2+t^2)}.$$

Thus, we can rewrite our difference of functions as

$$L(b,a) - L(a,b) = \int_0^{\infty} \frac{\frac{a^2}{b^2+t^2} - \frac{b^2}{b^2+t^2}}{\sqrt{(t^2+a^2)(t^2+b^2)}} dt = \int_0^{\infty} \frac{(a^2-b^2)t^2}{(t^2+a^2)^{\frac{3}{2}}(t^2+b^2)^{\frac{3}{2}}} dt.$$

Now, set $x = \frac{1}{2}\left(t - \frac{ab}{t}\right)$ and let

$$f(x) = \frac{(t^2+a^2)(t^2+b^2)}{t^2}.$$

Substituting in x for our expression ($dt = \frac{t}{\sqrt{x^2+ab}} dx$), we have

$$\int_0^{\infty} \frac{(a^2-b^2)t^2}{(t^2+a^2)^{\frac{3}{2}}(t^2+b^2)^{\frac{3}{2}}} dt = \int_{-\infty}^{\infty} \frac{(a^2-b^2)t^2}{(t^2+a^2)^{\frac{3}{2}}(t^2+b^2)^{\frac{3}{2}}} \cdot \frac{t}{\sqrt{x^2+ab}} dx = \int_{-\infty}^{\infty} \frac{(a^2-b^2)\,dx}{f(x)^{\frac{3}{2}} \cdot \sqrt{x^2+ab}}.$$

Now we have

$$f(x) = \frac{(t^2+a^2)(t^2+b^2)}{t^2} = t^2+a^2+b^2+\frac{a^2b^2}{t^2} = t^2-2ab+\frac{a^2b^2}{t^2}+a^2+2ab+b^2 = (2x)^2+(a+b)^2.$$

Thus (letting $a_1 = \frac{a+b}{2}$ and $b_1 = \sqrt{ab}$),

$$L(b,a) - L(a,b) = \int_{-\infty}^{\infty} \frac{(a^2-b^2)\,dx}{\left((2x)^2 + (a+b)^2\right)^{\frac{3}{2}} \cdot \sqrt{x^2+ab}} =$$

$$\left(a^2-b^2\right) \int_{-\infty}^{\infty} \frac{dx}{8\left(x^2 + \left(\frac{a+b}{2}\right)^2\right)^{\frac{3}{2}} \cdot \sqrt{x^2+ab}} =$$

$$\frac{a^2-b^2}{8} \int_{-\infty}^{\infty} \frac{dx}{(x^2+a_1^2)^{\frac{3}{2}} \cdot \sqrt{x^2+b_1^2}} =$$

$$\frac{a^2-b^2}{8a_1^2} \int_{-\infty}^{\infty} \frac{\frac{a_1^2}{a_1^2+x^2}dx}{(x^2+a_1^2)^{\frac{1}{2}} \cdot \sqrt{x^2+b_1^2}} =$$

$$\frac{a^2-b^2}{8a_1^2} \cdot 2 \int_{0}^{\infty} \frac{\frac{a_1^2}{a_1^2+x^2}dx}{\sqrt{(x^2+a_1^2)\cdot(x^2+b_1^2)}} =$$

$$\frac{a^2-b^2}{4a_1^2} \cdot L(b_1, a_1) = \frac{(a-b)(a+b)}{(a+b)^2} \cdot L(b_1,a_1) = \frac{a-b}{a+b} \cdot L(b_1,a_1).$$

Thus we have proven that

$$L(b,a) - L(a,b) = \frac{a-b}{a+b} \cdot L(b_1,a_1).$$

□

Proposition 3.7. Denote S as the following:

$$\sum_{j=1}^{\infty} 2^j \cdot c_j^2$$

where $c_n^2 = a_n^2 - b_n^2$. Then we have

$$2c_0^2 L(a,b) = \left(c_0^2 - S\right) I(a,b).$$

Proof. We have $4\left(a_1^2 - b_1^2\right) = 4\left(\frac{a+b}{2}\right)^2 - 4ab = (a+b)^2 - 4ab = (a-b)^2$, which can be used in the simplification of this expression (while using the results of Proposition 2.6):

$$4\left(a_1^2 - b_1^2\right) L(b_1,a_1) =$$

$$(a-b)^2 L(b_1,a_1) =$$

$$\left(a^2-b^2\right) \cdot \frac{a-b}{a+b} \cdot L(b_1,a_1) =$$

$$\left(a^2-b^2\right) \cdot \left(L(b,a) - L(a,b)\right) =$$

$$\left(a^2-b^2\right) \cdot \left(L(b,a) - \left(I(a,b) - L(b,a)\right)\right) =$$

$$\left(a^2-b^2\right)\cdot\left(2L\left(b,a\right)-I\left(a,b\right)\right).$$

Rewriting $a_1^2-b_1^2$ as c_1^2 and a^2-b^2 as c_0^2, we have

$$4c_1^2\cdot L\left(b_1,a_1\right)=c_0^2\cdot\left(2L\left(b,a\right)-I\left(a,b\right)\right)$$

$$2c_0^2\cdot L\left(b,a\right)-4c_1^2\cdot L\left(b_1,a_1\right)=c_0^2\cdot I\left(a,b\right).$$

This also works for all $j\in\mathbb{N}$:

$$2c_j^2\cdot L\left(b_j,a_j\right)-4c_{j+1}^2\cdot L\left(b_{j+1},a_{j+1}\right)=c_j^2\cdot I\left(a_j,b_j\right).$$

Multiplying by 2^j and noting that $I\left(a_j,b_j\right)=I\left(a,b\right)$ from Theorem 3.3, we see that

$$2^{j+1}\cdot c_j^2\cdot L\left(b_j,a_j\right)-2^{j+2}\cdot c_{j+1}^2\cdot L\left(b_{j+1},a_{j+1}\right)=2^j\cdot c_j^2\cdot I\left(a,b\right).$$

Adding the equations for $0\le j\le n$ gives us

$$\sum_{j=0}^{n}2^{j+1}\cdot c_j^2\cdot L\left(b_j,a_j\right)-\sum_{j=0}^{n}2^{j+2}\cdot c_{j+1}^2\cdot L\left(b_{j+1},a_{j+1}\right)=\sum_{j=0}^{n}2^j\cdot c_j^2\cdot I\left(a,b\right).$$

By shifting the index $k=j+1$ of the second sum we obtain

$$\sum_{j=0}^{n}2^{j+2}\cdot c_{j+1}^2\cdot L\left(b_{j+1},a_{j+1}\right)=\sum_{k=1}^{n+1}2^{k+1}\cdot c_k^2\cdot L\left(b_k,a_k\right).$$

Now we can cancel terms by telescoping our previous equation:

$$\sum_{j=0}^{n}2^{j+1}\cdot c_j^2\cdot L\left(b_j,a_j\right)-\sum_{j=0}^{n}2^{j+2}\cdot c_{j+1}^2\cdot L\left(b_{j+1},a_{j+1}\right)=$$

$$\sum_{j=0}^{n}2^{j+1}\cdot c_j^2\cdot L\left(b_j,a_j\right)-\sum_{k=1}^{n+1}2^{k+1}\cdot c_k^2\cdot L\left(b_k,a_k\right)=\sum_{j=0}^{n}2^j\cdot c_j^2\cdot I\left(a,b\right)$$

$$2^{0+1}\cdot c_0^2\cdot L\left(b_0,a_0\right)-2^{n+2}\cdot c_{n+1}^2\cdot L\left(b_{n+1},a_{n+1}\right)=\sum_{j=0}^{n}2^j\cdot c_j^2\cdot I\left(a,b\right).$$

Now we observe that

$$c_{n+1}^2=a_{n+1}^2-b_{n+1}^2=$$

$$\left(\frac{a_n+b_n}{2}\right)^2-a_nb_n=$$

$$\frac{a_n^2+2a_nb_n+b_n^2-4a_nb_n}{4}=$$

$$\frac{\left(a_n-b_n\right)^2}{4}=\frac{a_n-b_n}{4\left(a_n+b_n\right)}\cdot\left(a_n^2-b_n^2\right)=\frac{a_n-b_n}{4\left(a_n+b_n\right)}\cdot c_n^2.$$

The final result is less than $\frac{c_n^2}{4}$ because $b_n>0\rightarrow\frac{a_n-b_n}{a_n+b_n}<1$, and thus we have

$$c_{n+1}^2 < \frac{c_n^2}{4}.$$

We can extend this to see how $c_{n+1}^2 < 4^{-n-1}c_0^2$. Additionally, since $L\left(b_{n+1}, a_{n+1}\right) + L\left(a_{n+1}, b_{n+1}\right) = I\left(b_{n+1}, a_{n+1}\right)$ from Proposition 3.6, and both values on the left hand side are positive, we have

$$L\left(b_{n+1}, a_{n+1}\right) < I\left(b_{n+1}, a_{n+1}\right) = I\left(b, a\right).$$

Therefore, we have

$$2^{n+2} \cdot c_{n+1}^2 \cdot L\left(b_{n+1}, a_{n+1}\right) < 2^{n+2} \cdot 4^{-n-1} \cdot c_0^2 \cdot I\left(b, a\right) = 2^{-n} \cdot c_0^2 \cdot I\left(b, a\right).$$

Therefore, for $n \to \infty$ we lose our second term (it tends towards 0) and our equation becomes

$$2^{0+1} \cdot c_0^2 \cdot L\left(b_0, a_0\right) - \lim_{n\to\infty} \left(2^{n+2} \cdot c_{n+1}^2 \cdot L\left(b_{n+1}, a_{n+1}\right)\right) = \sum_{j=0}^{\infty} 2^j \cdot c_j^2 \cdot I\left(a, b\right)$$

$$2 \cdot c_0^2 \cdot L\left(b, a\right) = \sum_{j=0}^{\infty} 2^j \cdot c_j^2 \cdot I\left(a, b\right) = \left(c_0^2 + S\right) \cdot I\left(a, b\right).$$

Finally, using Proposition 3.6's $L\left(b, a\right) = I\left(a, b\right) - L\left(a, b\right)$ once again we have

$$2 \cdot c_0^2 \cdot L\left(b, a\right) = \left(c_0^2 + S\right) \cdot I\left(a, b\right)$$

$$2 \cdot c_0^2 \cdot \left(I\left(a, b\right) - L\left(a, b\right)\right) = \left(c_0^2 + S\right) \cdot I\left(a, b\right)$$

$$2 \cdot c_0^2 \cdot I\left(a, b\right) - \left(c_0^2 + S\right) \cdot I\left(a, b\right) = 2 \cdot c_0^2 \cdot L\left(a, b\right)$$

$$\left(c_0^2 - S\right) \cdot I\left(a, b\right) = 2 \cdot c_0^2 \cdot L\left(a, b\right).$$

□

Proposition 3.8. The following functions and their properties will prove very useful to us in the proof of Theorem 3.5:

The *Gamma Function* $\Gamma(x) = \int_0^\infty t^{x-1} \cdot e^{-t} dt$ satisfies for Re(x)> 0:

$$\Gamma(x+1) = x \cdot \Gamma(x)$$

and

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

The *Beta Function* $B\left(u, v\right) = \int_0^1 t^{u-1} \cdot \left(1-t\right)^{v-1} dt$ satisfies for Re(u)> 0 and Re(v)> 0:

$$B\left(u, v\right) = \frac{\Gamma(u) \cdot \Gamma(v)}{\Gamma(u+v)}.$$

These proofs involve matrices and integration by parts, something we are not very interested in right now. As a result, we will be skipping the proofs.
□

Proposition 3.9.

$$L\left(\sqrt{2},1\right)\cdot I\left(\sqrt{2},1\right)=\frac{\pi}{4}.$$

Proof. We notice the following due to the property $\sin^2+\cos^2=1$:

$$L\left(\sqrt{2},1\right)=\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta d\theta}{\sqrt{2\cos^2\theta+\sin^2\theta}}=\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta d\theta}{\sqrt{1+\cos^2\theta}}$$
$$I\left(\sqrt{2},1\right)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{2\cos^2\theta+\sin^2\theta}}=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1+\cos^2\theta}}$$

Next, if we substitute $x=\cos\theta$, we have $\theta=\arccos x\to d\theta=\frac{-dx}{\sqrt{1-x^2}}$:

$$L\left(\sqrt{2},1\right)=\int_0^{\frac{\pi}{2}}\frac{\cos^2\theta d\theta}{\sqrt{1+\cos^2\theta}}=\int_1^0\frac{x^2}{\sqrt{1+x^2}}\cdot\frac{-dx}{\sqrt{1-x^2}}=\int_0^1\frac{x^2dx}{\sqrt{1-x^4}}$$
$$I\left(\sqrt{2},1\right)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1+\cos^2\theta}}=\int_1^0\frac{1}{\sqrt{1+x^2}}\cdot\frac{-dx}{\sqrt{1-x^2}}=\int_0^1\frac{dx}{\sqrt{1-x^4}}$$

Now for another substitution, this time $x=t^{\frac{1}{4}}$. Then $dx=\frac{1}{4}t^{-\frac{3}{4}}dt$:

$$L\left(\sqrt{2},1\right)=\int_0^1\frac{x^2dx}{\sqrt{1-x^4}}=\int_0^1\frac{t^{\frac{1}{2}}}{\sqrt{1-t}}\cdot\frac{dt}{4t^{\frac{3}{4}}}=\int_0^1\frac{1}{4}\cdot t^{\frac{3}{4}-1}\cdot(1-t)^{\frac{1}{2}-1}\,dt=\frac{1}{4}\cdot B\left(\frac{3}{4},\frac{1}{2}\right).$$

$$I\left(\sqrt{2},1\right)=\int_0^1\frac{dx}{\sqrt{1-x^4}}=\int_0^1\frac{1}{\sqrt{1-t}}\cdot\frac{dt}{4t^{\frac{3}{4}}}=\int_0^1\frac{1}{4}\cdot t^{\frac{1}{4}-1}\cdot(1-t)^{\frac{1}{2}-1}\,dt=\frac{1}{4}\cdot B\left(\frac{1}{4},\frac{1}{2}\right).$$

Now with the contents of Proposition 3.8, we obtain the following:

$$L\left(\sqrt{2},1\right)\cdot I\left(\sqrt{2},1\right)=\frac{1}{4}\cdot B\left(\frac{3}{4},\frac{1}{2}\right)\cdot\frac{1}{4}\cdot B\left(\frac{1}{4},\frac{1}{2}\right)=\frac{1}{4}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}\cdot\frac{1}{4}\cdot\frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)}=$$

$$\frac{1}{16}\cdot\frac{\Gamma\left(\frac{1}{2}\right)^2\Gamma\left(\frac{1}{4}\right)}{\Gamma\left(\frac{5}{4}\right)}=\frac{1}{16}\cdot\frac{\pi\Gamma\left(\frac{1}{4}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)}=\frac{\pi}{4}.$$

□

Now we finally can prove our theorem with the aid of all the propositions we have.

Proof of Theorem 3.5. Note that both functions L and I satisfy the following property:

$$I\left(ma,mb\right)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{m^2a^2\cos^2\theta+m^2b^2\sin^2\theta}}=\frac{1}{|m|}\cdot I\left(a,b\right).$$

We can use this property to see why $I\left(1, \frac{1}{\sqrt{2}}\right) = \sqrt{2} \cdot I\left(\sqrt{2}, 1\right)$ and $L\left(1, \frac{1}{\sqrt{2}}\right) = \sqrt{2} \cdot L\left(\sqrt{2}, 1\right)$. Multiplying together and applying Proposition 3.9 gives us

$$I\left(1, \frac{1}{\sqrt{2}}\right) \cdot L\left(1, \frac{1}{\sqrt{2}}\right) = \sqrt{2} \cdot I\left(\sqrt{2}, 1\right) \cdot \sqrt{2} \cdot L\left(\sqrt{2}, 1\right) = 2 \cdot I\left(\sqrt{2}, 1\right) \cdot L\left(\sqrt{2}, 1\right) = 2 \cdot \frac{\pi}{4} = \frac{\pi}{2}.$$

Next, we use Proposition 3.7:

$$\left(c_0^2 - S\right) \cdot I\left(a, b\right) = 2 \cdot c_0^2 \cdot L\left(a, b\right)$$

$$\left(c_0^2 - S\right) \cdot I\left(a, b\right)^2 = 2 \cdot c_0^2 \cdot L\left(a, b\right) \cdot I\left(a, b\right).$$

Plugging in $a = 1$ and $b = \frac{1}{\sqrt{2}}$ and noting that $c_0^2 = 1^2 - \frac{1}{\sqrt{2}}^2 = \frac{1}{2}$, we have

$$\left(\frac{1}{2} - S\right) \cdot I\left(1, \frac{1}{\sqrt{2}}\right)^2 = 2 \cdot \frac{1}{2} \cdot L\left(1, \frac{1}{\sqrt{2}}\right) \cdot I\left(1, \frac{1}{\sqrt{2}}\right) = 2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{2}.$$

Remembering that we proved in Theorem 3.3 that

$$I\left(a, b\right) = \frac{\pi}{2 \cdot M\left(a, b\right)},$$

we get the following:

$$\left(\frac{1}{2} - S\right) \cdot I\left(1, \frac{1}{\sqrt{2}}\right)^2 = \frac{\pi}{2}$$

$$\left(\frac{1}{2} - S\right) \cdot \frac{\pi}{2 \cdot M\left(1, \frac{1}{\sqrt{2}}\right)}^2 = \frac{\pi}{2}.$$

We can rearrange a bit and also note that $S = \sum_{j=1}^{\infty} 2^j \cdot c_j^2$:

$$\left(\frac{1}{2} - S\right) \cdot \frac{\pi^2}{4 \cdot M\left(1, \frac{1}{\sqrt{2}}\right)^2} = \frac{\pi}{2}$$

$$\left(\frac{1}{2} - S\right) \cdot \frac{\pi}{2 \cdot M\left(1, \frac{1}{\sqrt{2}}\right)^2} = 1$$

$$\left(\frac{1}{2} - S\right) \cdot \pi = 2 \cdot M\left(1, \frac{1}{\sqrt{2}}\right)^2$$

$$\left(1 - 2S\right) \cdot \pi = 4 \cdot M\left(1, \frac{1}{\sqrt{2}}\right)^2$$

$$\pi = \frac{4 \cdot M\left(1, \frac{1}{\sqrt{2}}\right)^2}{\left(1 - 2 \cdot \sum_{j=1}^{\infty} 2^j \cdot c_j^2\right)}.$$

□

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