Alternating Sign Matrices

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Alternating Sign Matrices

Alternating sign matrices appear when using Dodgson condensation to calculate a determinant. Additionally, they are related to the six vertex model in statistical mechanics.

Definition. An *alternating sign matrix* is an $n \times n$ matrix with entries of $0, 1$, or -1 such that

- the sum of a row or column is equal to 1,
- the nonzero entries of the rows and columns alternate signs.

For example,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a 4×4 alternating sign matrix.

Descending Plane Partitions

Definition. A *descending plane partition* is a plane partition with

- rows that do not increase from left to right,
- columns that decrease downwards,
- a number of entries in each row strictly less than the largest entry in the row,
- indented left side edges.

Notably, there are

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}$$

descending plane partitions of order n.

Alternating Sign Matrix Theorem

***Theorem* 1.** The number of $n \times n$ alternating sign matrices is given by

$$\prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

***Theorem* 2.** The number of $n \times n$ alternating sign matrices with a unique 1 in the first row in the *ith* column is given by

$$\frac{\binom{n+i-2}{n-1}\binom{2n-i-1}{n-1}}{\binom{3n-2}{n-1}} \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

Definition. A *monotone triangle* of order n is a triangle with n entries along the sides and base with the entries between 1 and n such that

- entries strictly increase from left to right across rows,
- entries increase diagonally towards right.

Let the ith row of the triangle equal the positions of 1s in the sum of the first i rows of an alternating sign matrix.

Example

For example,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then,

$$r_1 = (1,0,0) = 1$$

$$r_1 + r_2 = (1,0,0) + (0,0,1) = 1,3$$

$$r_1 + r_2 + r_3 = (1,0,0) + (0,0,1) + (0,1,0) = 1,2,3.$$

With that,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$\longleftrightarrow$

$$\begin{matrix} & & 1 & & \\ & 1 & & 3 & \\ 1 & & 2 & & 3. \end{matrix}$$

Definition. The *shift operator* E_x is

$$E_x p(x) = p(x+1)$$

and

$$\mathrm{S}_{x,y} = E_x + E_y^{-1} - E_x E_y^{-1}.$$

Additionally, the *forward difference* and *backward difference* are

$$\overline{\Delta_x} = E_x - Id$$

and

$$\underline{\Delta_x} = Id - E_x^{-1}$$

respectively.

Additionally, let

$$\mathrm{GT}_n(\mathbf{x}) = \prod_{1\le i<j\le n} \frac{x_i - x_j + j - i}{j-i},$$

$$M_n(\mathbf{x}) = \prod_{1\le p<q\le n} \mathrm{S}_{x_q,x_p} \mathrm{GT}_n(\mathbf{x}),$$

and

$$\mathrm{MT}_\lambda = \sum_{\substack{\mu\prec\lambda \\ \mu S}} \mathrm{MT}_\mu = \text{ The number of monotone triangles with bottom row } \lambda.$$

Theorem 3. Let $n \ge 2$. Suppose $P(\mathbf{x}), Q(\mathbf{x})$ are polynomials in $\mathbf{x} = (x_1, ..., x_{n-1})$ with $P(\mathbf{x}) = \overline{\Delta_{\mathbf{x}}} Q(\mathbf{x})$. Furthermore, suppose if $x_{i+1} = x_i + 1$, then for every $i = 1, 2, ..., n-2$ $\mathrm{S}_{x_i,x_{i+1}} Q(\mathbf{x})$ vanishes. If λ is a partition with n parts, then

$$\sum_{\substack{\mu\prec\lambda \\ \mu strict}} P(\mu) = \sum_{r=1}^{n} (-1)^{r+n} Q(\lambda_1+1, ..., \lambda_{r-1}+1, \lambda_{r+1}, .., \lambda_n).$$

Theorem 4. Let $d_1, d_2, .., d_{n-1} \ge 0$ be integers with $d_n = -1$. If λ is a partition with n parts, then

$$\sum_{\substack{\mu\prec\lambda \\ \mu strict}} \prod_{1\le p<q\le n-1} \mathrm{S}_{\mu_q,\mu_p} \det_{1\le i,j\le n-1} \binom{\mu_i - i + n - 1}{d_j}$$

$$= \prod_{1\le p<q\le n} \mathrm{S}_{\lambda_1,\lambda_p} \det_{1\le p,q\le n} \binom{\lambda_i - i + n}{d_j + 1}.$$

Theorem 5. Let $\lambda = (\lambda_1, ..., \lambda_n)$ be a strict partition, then the number of monotone triangles with bottom row λ is $M_n(\mathbf{x})$ at $(x_1, ..., x_n) = (\lambda_1, ..., \lambda_n)$.

This follows from 3 and 4.

Definition.

$$\mathrm{rot}(\lambda) = (\lambda_n - n, \lambda_1, ..., \lambda_{n-1}).$$

***Theorem* 6.** Let $n \geq 1$ and $1 \leq r \leq n$. Then,

$$e_r(\overline{\Delta_{x_1}}, .., \overline{\Delta_{x_n}}) \prod_{1 \leq i < j \leq n} \frac{x_i - x_j + j - i}{j - i}$$

$$= e_r(\underline{\Delta_{x_1}}, .., \underline{\Delta_{x_n}}) \prod_{1 \leq i < j \leq n} \frac{x_i - x_j + j - i}{j - i} = 0.$$

Proof. Considering

$$E_{x_1} E_{x_2}^2 ... E_{x_n}^n e_r(\overline{\Delta_{x_1}}, ..., \overline{\Delta_{x_n}}) \prod_{1 \leq i < j \leq n} \frac{x_i - x_j + j - i}{j - i}$$

$$= e_r(\overline{\Delta_{x_1}}, ..., \overline{\Delta_{x_n}}) \prod_{1 \leq i < j \leq n} \frac{x_i - x_j}{j - i},$$

it follows that the right-hand side vanishes. □

Suppose λ is an integer vector of length n, then

$$MT_\lambda = (-1)^{n-1} MT_{\mathrm{rot}(\lambda)}.$$

***Theorem* 7.**

- (1) The number of monotone triangles with bottom row $1, 2, ..., n$ and i occurrences of 1 is equal to the evaluation of the polynomial $(-\overline{\Delta}_{x_n})^{i-1} M_n(x_1, ..., x_n)at(x_1, ..., x_n) = (n, n-1, ..., 3, 2, 2)$.
- (2) The number of monotone triangles with bottom row $1, 2, ..., n$ and i occurrences of n is equal to the evaluation of the polynomial $\underline{\Delta}_{x_1}^{i-1} M_n(x_1, ..., x_n)at(x_1, ..., x_n) = (n-1, n-1, n-2, ..., 2, 1)$.

***Theorem* 8.** Let $n \geq 1$. Then,

$$A_{n,i} = \sum_{j=1}^{n} \binom{2n - i - 1}{n - i - j + 1} (-1)^{j+1} A_{n,j}, i = 1, 2, ..., n.$$

Proof. Using 7(1),

$$A_{n,1} = (-1)^{n+i} \overline{\Delta}_{x_n}^{i-1} M_n(x_n - n, n-1, ..., 2)|_{x_n = 2},$$

which is equal to

$$(-1)^{n+i} (Id - \underline{\Delta}_{x_n}^{i-1} M_n(x_n, n-1, n-2, .., 1)|_{x_n = n-1}.$$

Then, using 7(2),

$$\sum_{j \geq 0} \binom{2n - i - 1}{j} (-1)^{n+i+j} A_{n,i+j}$$

$$= \sum_{j=1}^{n} \binom{2n - i - 1}{j - i} (-1)^{n+j} A_{n,j}$$

where $A_n, j = 0$ when $j > n$.

□