Rational Functions, Recurrences, and Explicit Formulas

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1 Introduction

In this paper, we cover the relationship between rational generating functions, linear recurrences, and explicit formulas. We prove a theorem about this through vector spaces. We then show some corollaries of the theorem.

We will assume knowledge of vector spaces, the Fundamental Theorem of Algebra, and the Generalized Binomial Theorem.

2 Main Theorem on Equivalent Representations

Definition 2.1. A rational function over $\mathbb{C}$ is a function of the form $P(x)/Q(x)$ where P and Q are polynomials in x over $\mathbb{C}$.

Example. The classic sequence which has a rational generating function is the Fibonacci numbers, defined by $f_0 = 0, f_1 = 1$, and the recurrence $f_n = f_{n-1} + f_{n-2}$. Let $F(x) = \sum_{n=0}^{\infty} f_n x^n$. We have that $(x + x^2)F(x) = \sum_{n=2}^{\infty}(f_{n-2} + f_{n-1})x^n = \sum_{n=2}^{\infty} f_n x^n = F(x) - x$. Thus, $(1 - x - x^2)F(x) = x$, so $F(x) = \frac{x}{1-x-x^2}$. Expanding with partial fractions, and letting $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$, we have $F(x) = \frac{1}{\sqrt{5}}\frac{1}{1-\varphi x} - \frac{1}{\sqrt{5}}\frac{1}{1-\psi x}$. Expanding as geometric series and taking the x^n coefficient of both sides, we get the classic result $f_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)$.

This example demonstrates the connections between recurrences, rational generating functions, and explicit formulas, which are more fully outlined in the following theorem.

Theorem 2.2. *Let Q be a fixed polynomial of degree d, say $1+c_1x+c_2x^2+\cdots+c_dx^d$ (we assume the constant term is 1 without loss of generality). The following are equivalent conditions on the sequence $(a_n)_{n\in\mathbb{N}}$:*

1. *The generating function of $(a_n)_{n\in\mathbb{N}}$, $\sum_{n=0}^{\infty} a_n x^n$, is a rational function $\frac{P(x)}{Q(x)}$ where P has lesser degree than Q.*

2. *The sequence $(a_n)_{n\in\mathbb{N}}$ satisfies the recurrence*

$$a_n + c_1 a_{n-1} + \cdots + c_d a_{n-d} = 0$$

   for all $n \geq d$.

3. *Let r_i be the roots of Q, with multiplicity d_i respectively. Say there are k roots. The sequence $(a_n)_{n\in\mathbb{N}}$ satisfies the explicit formula*

$$a_n = \sum_{i=1}^{k} P_i(n)(\frac{1}{r_i})^n,$$

where each for each i, P_i is a polynomial with degree less than d_i.

Proof. Fix Q. Say it has degree d. Let V_1, V_2, and V_3 be the sets of sequences of complex numbers which satisfy each of our three conditions respectively for our given Q.

Lemma 2.3. *V_1, V_2 and V_3 are vector spaces over $\mathbb{C}$ under the standard definitions of addition and scalar multiplication of sequences:$(x_n)_{n\in\mathbb{N}}+(y_n)_{n\in\mathbb{N}} = (x_n+y_n)_{n\in\mathbb{N}}$ and $c(x_n)_{n\in\mathbb{N}} = (cx_n)_{n\in\mathbb{N}}$. Furthermore V_1 and V_2 have dimension d and V_3 has dimension at most d.*

Proof. Let $(a_n)_{n\in\mathbb{N}}$, $(b_n)_{n\in\mathbb{N}}$ be arbitrary sequences in $\mathbb{C}$ and let c be an arbitrary constant. We show that each set is closed under addition and scalar multiplication, which will establish it is a subspace of $\mathbb{C}^{\mathbb{N}}$.

1. Suppose $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in V_1$. By definition, $\sum_{n=0}^{\infty} a_n x^n = \frac{P_a(x)}{Q(x)}$ and $\sum_{n=0}^{\infty} b_n x^n = \frac{P_b(x)}{Q(x)}$ for some polynomials P_a and P_b with degree less than d. We have

$$\sum_{n=0}^{\infty}(a_n + b_n)x^n = \sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \frac{P_a(x) + P_b(x)}{Q(x)}.$$

   Because $P_a + P_b$ has degree less than d, $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}}$ is in V_1 as desired. Similarly,

$$\sum_{n=0}^{\infty}(ca_n)x^n = c\sum_{n=0}^{\infty} a_n x^n = \frac{cP_a(x)}{Q(x)}.$$

   We have cP_a has degree less than d, so $c(a_n)_{n\in\mathbb{N}} \in V_1$.

   Note that $(a_n)_{n\in\mathbb{N}}$ is isomorphic to the vector space of polynomials with degree less than d under the mapping of $(a_n)_{n\in\mathbb{N}}$ by taking $P(x) = Q(x)\sum_{n=0}^{\infty} a_n x^n$, and the inverse mapping of expanding and taking the coefficients of $\frac{P(x)}{Q(x)}$. Polynomials of degree less than d are spanned by the basis of $1, x, x^2, \cdots, x^{d-1}$, and thus have dimension d. We have thus shown V_1 is a vector space of dimension d.

2. Suppose $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in V_2$. By definition, $a_n + c_1 a_{n-1} + \cdots + c_d a_{n-d} = 0$ and $b_n + c_1 b_{n-1} + \cdots + b_d a_{n-d} = 0$ for $n \geq d$. We have $(a_n+b_n)+c_1(a_{n-1}+b_{n-1})+\cdots+c_d(a_{n-d}+b_{n-d})$ $= (a_n+c_1a_{n-1}+\cdots+c_da_{n-d})+(b_n+c_1b_{n-1}+\cdots+c_db_{n-d}) = 0$ as desired, so $(a_n)_{n\in\mathbb{N}}+(b_n)_{n\in\mathbb{N}}$ is in V_2. Furthermore $(ca_n)+c_1(ca_{n-1})+\cdots+c_d(ca_{n-d}) = c(a_n+c_1a_{n-1}+\cdots+c_da_{n-d}) = 0$, so $c(a_n)_{n\in\mathbb{N}}$ is also in V_2

   For any sequence in V_2, selecting values for $a_0, a_1, \cdots, a_{d-1}$ uniquely determines the rest of the sequence by the recurrence. For $0 \leq i \leq d-1$, let $(s_n^{(i)})_{n\in\mathbb{N}}$ be the sequence where $s_i^{(i)} = 1$ and $s_j^{(i)} = 0$ for $0 \leq j \leq d-1$ where $j \neq i$. Any $(a_n)_{n\in\mathbb{N}}$ can be uniquely written as a linear combination of the $s^{(i)}$'s with $(a_n)_{n\in\mathbb{N}} = \sum_{i=0}^{d-1} a_i(s_n^{(i)})_{n\in\mathbb{N}}$. This is the only linear combination where the first d terms are the same (which ensures that all the subsequent terms are equal). Thus $(s_n^{(i)})_{n\in\mathbb{N}}$ for $0 \leq i \leq d-1$ forms a basis.

   Consequently V_2 is a vector space of dimension d.

3. Suppose $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in V_3$. Let r_i be the roots of Q with multiplicity d_i respectively. Say there are k of them. By definition, $a_n = \sum_{i=1}^k P_i(n)(\frac{1}{r_i})^n$ and $b_n = \sum_{i=1}^k P_i'(n)(\frac{1}{r_i})^n$, where for each i, P_i and P_i' are polynomials with degree less than d_i. Then, $a_n + b_n = \sum_{i=1}^k P_i(n)(\frac{1}{r_i})^n + \sum_{i=1}^k P_i'(n)(\frac{1}{r_i})^n = \sum_{i=1}^k (P_i(n)+P_i'(n))(\frac{1}{r_i})^n$. Because for each i, $P_i + P_i'$ is a polynomial with degree less than d_i, we have that $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} \in V_3$. Similarly, $ca_n = c\sum_{i=1}^k P_i(n)(\frac{1}{r_i})^n = \sum_{i=1}^k cP_i(n)(\frac{1}{r_i})^n$. We have for each i, that cP_i has degree less than d_i, so $c(a_n)_{n\in\mathbb{N}} \in V_3$.

   The sequences $(n^j(\frac{1}{r_i})^n)_{n\in\mathbb{N}}$, where $1 \le i \le k$ and $0 \le j \le d_i - 1$, spans V_3. There are d_i spanning vectors for each root r_i, and $\sum_{i=1}^k d_i = d$, so there are d spanning vectors, so the dimension of V_3 is at most d (in fact we will see that it is d but we will show that later).

Let V_4 be the set of sequences $(a_n)_{n\in\mathbb{N}}$ such that $\sum_{n=0}^\infty a_n x^n = \sum_{i=1}^k \sum_{j=1}^{d_i} c_{ij}(1 - \frac{1}{r_i}x)^{-j}$ for some complex c_{ij}'s (which can be motivated as the sequences which have a generating function that admits a partial fraction decomposition). We quickly verify this is a vector space.

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ be sequences of complex numbers which are in V_4 and let c be a complex constant. We have that $\sum_{n=0}^\infty a_n x^n = \sum_{i=1}^k \sum_{j=1}^{d_i} c_{ij}(1 - \frac{1}{r_i}x)^{-j}$ and $\sum_{n=0}^\infty b_n x^n = \sum_{i=1}^k \sum_{j=1}^{d_i} c_{ij}'(1 - \frac{1}{r_i}x)^{-j}$ by definition for some constants c_{ij} and c_{ij}'. We have $\sum_{n=0}^\infty (a_n + b_n)x^n = \sum_{i=1}^k \sum_{j=1}^{d_i} (c_{ij} + c_{ij}')(1 - \frac{1}{r_i}x)^{-j}$, so $(a_n)_{n\in\mathbb{N}} + (b_n)_{n\in\mathbb{N}} \in V_4$. Similarly, $\sum_{n=0}^\infty (ca_n)x^n \sum_{i=1}^k \sum_{j=1}^{d_i} c \cdot c_{ij}(1 - \frac{1}{r_i}x)^{-j}$, so $c(a_n)_{n\in\mathbb{N}} \in V_4$.

Let $R_{ij}(x) = (1 - \frac{1}{r_i}x)^{-j}$ for $1 \le i \le k$ and $1 \le j \le d_i$. This is a set of d functions ($\sum_{i=1}^k d_i = d$) which spans the generating functions of the sequences. We now verify the functions are linearly independent (which will show that their corresponding sequences are also linearly independent). Suppose for the sake of contradiction that we have $\sum c_{ij} R_{ij}(x) = 0$, for constants c_{ij} not identically 0. Take i' such that for some value of j, $c_{i'j} \ne 0$, and let j' be the highest value of j such that $c_{i'j} \ne 0$. Then, consider $\sum c_{ij} R_{ij}(x)(1 - \frac{1}{r_i'}x)^{j'}$ evaluated at r_i. Every term except $c_{i'j'} R_{ij}(x)(1 - \frac{1}{r_i'}x)^{j'} = c_{i'j'}$ will vanish, as it will have $(1 - \frac{1}{r_i'}x)$ in its product, so the sum equals $c_{i'j'}$. However, the sum evaluates to 0, so we conclude $c_{i'j'} = 0$, which is a contradiction. Thus our set is a linearly independent, and consequently a basis, proving that V_4 is a vector space of dimension d.

We will show the vector spaces V_1, V_2, V_3, and V_4 are equivalent to establish our theorem.

Consider an arbitrary basis vector of V_4, say the coefficients of $(1 - \frac{1}{r_i}x)^{-j}$. Expanding by the generalized binomial theorem, we get $\sum_{n=0}^\infty \binom{-j}{n}(-\frac{1}{r_i})^n x^n$, or the sequence $((-1)^n\binom{-j}{n}(\frac{1}{r_i})^n)_{n\in\mathbb{N}} = (\binom{j+n-1}{n}(\frac{1}{r_i})^n)_{n\in\mathbb{N}}$. Expanding in terms of n, we have that $\binom{j+n-1}{n} = \frac{(j+n-1)\cdots(n+1)}{(j-1)!}$ is a degree j polynomial in n. Thus the sequence is of the form $P_i(n)(\frac{1}{r_i})^n$, so any basis of V_4 is in V_3. We have $V_4 \subseteq V_3$. We thus have $d \ge \dim V_3 \ge \dim V_4 = d$, so V_3 has dimension d. If a vector space ant it's subspace have the same dimension, they must be equal, so $V_3 = V_4$.

Consider an arbitrary sequence in V_1. We then have that $Q(x)\sum_{n=0}^\infty a_n x^n = P(x)$. Considering the coefficient of x^n for $n \ge d$, we get that $a_n + c_1 a_{n-1} + \cdots + c_d a_{n-d} = 0$ as desired. Thus we have that $V_1 \subseteq V_2$, and $V_1 = V_2$.

Consider an arbitrary sequence in V_4. It has generating function $\sum_{i=1}^k \sum_{j=1}^{d_i} c_{ij}(1 - \frac{1}{r_i}x)^{-j}$ for some constants c_{ij} by definition. We can convert each of these fractions into the common denominator of $Q(x)$, and the numerator will have lesser degree. Thus, we have that $V_4 \subseteq V_1$, so $V_4 = V_1$.

Thus we have that $V_1 = V_2 = V_3$ as desired.

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Corollary 2.4. *Let Q be a fixed polynomial of degree d, say $1 + c_1x + c_2x^2 + \cdots + c_dx^d$. The following are equivalent conditions on the sequence $(a_n)_{n\in\mathbb{N}}$:*

1. *The generating function of $(a_n)_{n\in\mathbb{N}}$, $\sum_{n=0}^{\infty} a_nx^n$, is a rational function $\frac{P(x)}{Q(x)}$.*

2. *The sequence $(a_n)_{n\in\mathbb{N}}$ eventually satisfies the recurrence*

$$a_n + c_1a_{n-1} + \cdots + c_da_{n-d} = 0$$

.

3. *Let r_i be the roots of Q, with multiplicity d_i respectively. Say there are k roots. The sequence $(a_n)_{n\in\mathbb{N}}$ eventually satisfies the explicit formula*

$$a_n = \sum_{i=1}^{k} P_i(n)(\frac{1}{r_i})^n,$$

where each for each i, P_i is a polynomial with degree less than d_i.

Proof. Suppose $(a_n)_{n\in\mathbb{N}}$ has generating function $\frac{P(x)}{Q(x)}$. Through polynomial division, we get $\frac{P(x)}{Q(x)} = L(x) + \frac{R(x)}{Q(x)}$ where R is the remainder polynomial with degree less than Q. We have that $L(x)$ is a polynomial, which has some finite degree, so for sufficiently high x^n, the coefficients will equal those of $\frac{R(n)}{Q(n)}$ which we can apply our Theorem thus implying the recurrence and explicit formula eventually hold.

If a sequence eventually satisfies either the recurrence or the explicit formula, we can consider the sequence which fully satisfies the recurrence/explicit formula. By our original theorem, this has some rational function $\frac{P(x)}{Q(x)}$ where the degree of P is less than the degree Q. Because our generating function only differs in a finite number of terms, it can differ at most by some polynomial, say $L(x)$. Thus we have a rational generating function $L(x) + \frac{P(x)}{Q(x)}$. □

3 Applications

3.1 Hadamard product

Corollary 3.1. *If $(a_n)_{n\in\mathbb{N}}$ and $(b_n)_{n\in\mathbb{N}}$ are have rational generating functions, their Hadamard product, or $(a_nb_n)_{n\in\mathbb{N}}$ also has a rational generating function*

Proof. We have $a_n = \sum_{i=1}^{k} P_i(n)(\frac{1}{r_i})^n$ and $b_n = \sum_{i=1}^{k'} P_i'(n)\frac{1}{r_i'}^n$ for sufficiently large n by Corollary 3.1. Then $a_nb_n = \sum_{i=1}^{k} \sum j = 1^{k'} P_i(n)P_j'(n)(\frac{1}{r_i}\frac{1}{r_j'})^n$. We can construct a polynomial with roots r_ir_j' with the correct multiplicity as dictated by the degrees of the $P_i(n)P_j'(n)$'s. Then by Corollary 3.1, we have that $(a_nb_n)_{n\in\mathbb{N}}$ is rational. □

Example. We will find the generating function of f_n^2. We have $f_n = \frac{1}{\sqrt{5}}(\varphi^n - \psi^n)$, so

$$f_n^2 = \frac{1}{5}((\varphi^2)^n - 2(\varphi\psi)^n + (\psi^2)^n).$$

By Corollary 3.1, we have that this has a rational generating function with denominator $(x - \varphi^2)(x - \varphi\psi)(x - \psi^2) = x^3 - 2x^2 - 2x + 1$. We can find the numerator by expanding out terms of $(x^3 - 2x^2 - 2x + 1)\sum_{n=0}^{\infty} f_n^2 x^n$ to get the numerator is $x - x^2$, so the generating function is $\frac{x-x^2}{x^3-2x^2-2x+1}$.

3.2 Asymptotics of rational generating functions

Remark 3.2. The rational generating function can be used to find asymptotics of a_n. If a_n has a root of least modulus, say r_i, then in the explicit formula, the $P_i(n)(\frac{1}{r_i})^n$ term will dominate, giving the asymptotic $a_n \sim P_i(n)(\frac{1}{r_i})^n$.

Example. Consider the sequence a_n of integer partitions of n with summands from some fixed finite set of integers, say S, with $\gcd(S) = 1$. The generating function for this sequence, $\sum_{n=0}^{\infty} a_n x^n = \prod_{s\in\mathbb{S}}(1 + x^s + x^{2s} + x^{3s} + \cdots) = \prod_{s\in\mathbb{S}} \frac{1}{1-x^s}$. This is a rational function so we can use the techniques we have developed. All the roots are roots of unity, and 1 has multiplicity $|S|$. All the other roots have less multiplicity as no other root of unity can divide all the terms $1 - x^s$ by our assumption $\gcd(S) = 1$. Thus the term from the root at 1 of order $n^{|S|-1}$ will dominate the asymptotics. More careful consideration of the coefficients and partial fraction expansion, which we will not do here, will give us Schur's theorem which includes the constant factor:

$$a_n \sim \frac{1}{\prod_{s\in S} s} \frac{n^{|S|-1|}}{(|S|-1)!}.$$

The number of ways of breaking change for n cents is a problem of partitioning n into summands of the set $\{1, 5, 10, 25, 50\}$, so by Schur's theorem there are approximately $\frac{n^4}{1500000}$ ways.

Sources

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