COMBINATORIAL SPECIES

TANVI DESHPANDE

1. Combinatorial Species

Combinatorial species is a category-theoretic approach to generating functions. It was developed by Andre Joyal in 1981; his original paper on the subject gave a new, interesting proof of Cayley's theorem, a novel species proof of the Lagrange inversion formula, and redeveloped Polya's enumeration theory through a species lens. This paper will give the motivation for studying combinatorial species through illustrative examples, introduce the category theory background necessary to study species, and culminate in Joyal's proof of the Lagrange inversion formula. This paper uses a combination of results from the following works: [1] [2] [3].

Definition 1.1 (Combinatorial species)**.** Intuitively, a combinatorial species is a labelled structure constructed from a set A that does not depend on the elements of A. The species S is a function that sends the elements of A to the set of all structure of S built from A.

Example. For instance, we might have $A = \{1, 2, 3, 4\}$, and a species on A might be the trees built from A. Or, we might have $A = \{a, b, c, d, e\}$, and a species on A might be linear orders on A, or partitions of A.

The essence of combinatorial species really comes from the, as Joyal puts it, *transfer* of species: that is, "swapping" out the labels of a structure for another set of labels. Using category theory as a backbone for the concept of species makes this much easier to achieve than say, using only formal set theory, which also results in a cleaner definition of species than past ones.

We'll get to the category theory in a bit, but here's the formal definition:

Definition 1.2 (Combinatorial species, formally). A combinatorial species is a functor from $\mathbb{B} \to \mathbb{B}$, where $\mathbb{B}$ is the category whose objects are finite sets and morphisms are bijections.

Notation. For a finite set E, $M[E]$ is the set of structures of the species M on E, and we can say that E underlies $M[E]$. An element $S \in M[E]$ is said to be an M-structure.

This all might be made more clear by an example:

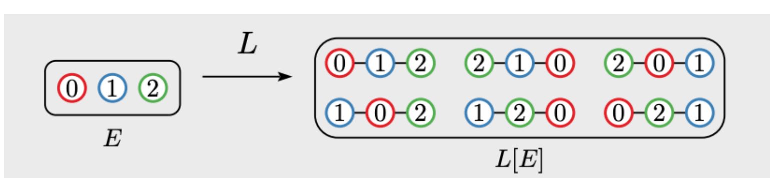

Figure 1

In Figure 1, L is the species of linear orders, $L[E]$ is the set of L-structures supported by E, and an element of $s \in L[E]$ is an L-structure, with $E = \{0, 1, 2\}$ as its underlying set. Note that the species of linear orders is *equipotent* to the species of permutations but not isomorphic.

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2. Category Theory Background

Category theory is the study of structures, in terms of labeled directed graphs called *categories*, which consist of objects and morphisms, or mappings, between the objects. Species, as given before, are functors, or mappings between categories.

Definition 2.1 (Category). Categories are the building blocks of category theory; they consist of objects, and mappings between these objects, such that each object has an identity mapping and mappings are associative (we can compose them).

The definition seems a little wishy-washy, but it can be understood better through some examples: A typical category is the category whose objects are finite sets and mappings are total functions (functions $f : A \to B$ where all f is defined for all $a \in A$); this category is denoted as $\mathbb{E}$.

Definition 2.2 (Functor). Since categories are mathematical objects, we can define mappings between them; these are called functors, and are denoted as arrows between categories ($\mathbb{B} \to \mathbb{E}$, for instance). Functors map objects from one category to objects of the other, and morphisms from one category to morphisms of the other.

Interestingly, this means that categories themselves form a category – the objects are categories, and the morphisms are functors. So do species – they are functors, and their morphisms are *natural transformations*.

Definition 2.3 (Natural Transformation). A natural transformation is a mapping between two functors F and G on the same categories (say, C and D), such that the *internal structure* of the categories is preserved. A morphism η has the following properties, the second of which are illustrated by the following diagram:

- Each object $X \in C$ is equipped with a morphism η_X that maps $F(X) \to G(X)$.
- For every morphism $f : X \to Y$ in C, $\eta_Y \circ F(f) = G(f) \circ \eta_X$.

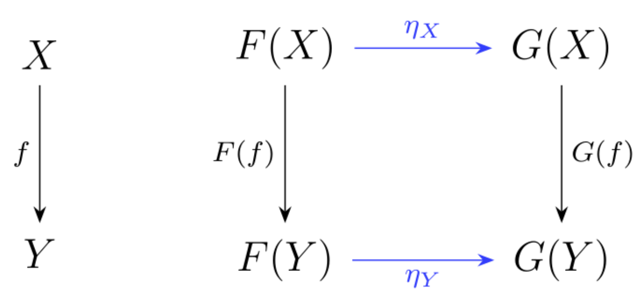

Figure 2

Figure 2 shows nicely that each of the paths taken from $F(X)$ to $G(Y)$ are equivalent. So, a functor is a morphism of categories, and a natural transformation is a morphism of functors.

Definition 2.4 (Groupoids). A groupoid is a category equipped with invertible morphisms: each map $m : A \to B$ also comes with a map $m^{-1} : B \to A$ such that $m^{-1} \circ m$ and $m \circ m^{-1}$ are the identity mappings.

For example, the category $\mathbb{B}$, of finite sets and bijections, which we have already seen, is a groupoid, since bijections are obviously invertible.

Returning to the definition of species, we can unpack it slightly. In Figure 1, the species L is a functor $\mathbb{B} \to \mathbb{B}$, mapping finite sets of labels to finite sets of linear orderings on the labels. Therefore the species is not exactly a combinatorial class or similar structure, but a mapping from the "pieces" of a structure to the assembled structure itself. A species is like a cookie cutter for creating structures of a certain kind for different underlying sets.

Definition 2.5. $S[n]$ is the set of S-structures built from $[n]$. A species S has the exponential generating function

$$S(x) = \sum_{n \geq 0} |S[n]| \frac{x^n}{n!}.$$

3. Operations on species and other cool stuff

The sum and product operations are defined nicely for species:

Definition 3.1 (Sum). The sum of two species M and N is defined as $(M + N)[E] = M(E) + N(E)$, a coproduct in category theory. A family of species $(M_i)_{i \in I}$ is summable if I, the set of indices where M_i is nonempty, is finite.

Products are a bit trickier to define.

Definition 3.2 (Product). The product of two species S and T is obtained by partitioning the underlying set E into two disjoint components E_1 and E_2, then putting an S-structure on E_1 and a T-structure on E_2. More formally,

$$(S \cdot T)(E) = \bigsqcup_{E} S(E_1) \times T(E_2).$$

Mercifully, these operations carry over nicely to generating functions: $(S + T)(x) = S(x) + T(x)$ and $(S \cdot T)(x) = S(x) \cdot T(x)$. Other definitions from generating functions also carry over nicely to the world of species:

Definition 3.3. The species X is the singleton species, consisting of just one object; it can be considered analogous to $\mathcal{Z}$ in our study of generating functions.

One interesting thing we can do is "lift" identities to the world of species. One silly example is

$$\frac{1}{1-x} = \frac{1-x+x}{1-x} = 1 + x \cdot \frac{1}{1-x}.$$

Because $L(x) = \frac{1}{1-x}$, we can write that

$$L \approx 1 + X \cdot L,$$

or combinatorially, "a linear order is either empty, or has a first element followed by a linear order." Lastly, we have assemblies and substitutions.

Definition 3.4 (Assembly). An N-assembly on a species E is a partition of E, with each component being equipped with an N-structure. The divided power $\gamma_n(N)$ is the species of N-assemblies with n parts, and the exponential $\exp(N)$ is the species of all N-assemblies.

Assemblies are basically sets – forests are assemblies of trees, permutations are assemblies of cycles, partitions are assemblies of integers, etc.

Definition 3.5 (Substitution). For species R and N, we can form the species $R(N)$, or an R-assembly. We do so by partitioning the underlying set E into k parts, placing N-structures on each of the k parts, and placing an R structure on the set of N-structures. Equivalently,

$$R(N(A)) = \bigsqcup_{E = \sqcup_{i \leq k} B_i} R[k] \times \prod_{i \leq k} N(E_i).$$

Once again, composition carries over to generating functions of species.

Definition 3.6 (Derivatives). The derivative species S' of a species S is:

$$S'[E] = S[E^+]$$

where E^+ is the set E plus an additional item: $E + \{*\}$.

An S'-structure is just an S-structure with a distinguished element: the species of linear orders is the derivative of the species of cycles.

Definition 3.7 (r-Enrichment)*.* For a combinatorial species R, we say that an endofunction $\phi : E \to E$ is R-enriched if each of its fibers $\phi^{-1}\{x\}, x \in E$, or the set of vertices of ϕ connected to x, is equipped with an r-structure.

Example. The notion of r-enriched trees comes about when we think of each of the fibers (or nodes) of graphical representations of endofunctions as having an r-structure corresponding to it.

Example. With our definitions of products, enrichment, and assemblies, we can see that A, the species of rooted trees, is equal to $X \cdot \exp(A)$. More generally, the A_R, the species of R-enriched rooted trees, is equal to $X \cdot R(A_R)$.

Proposition 3.8*. We have $D = S(A)$, where D, S, and A are the species of endofunctions, permutations, and rooted trees respectively.*

Proof. Consider an endofunction $\phi \in D[E]$. A point $x \in E$ is periodic if $\phi^n(x) = x$ for some n. For $x \in E$, let $v(x)$ be the first periodic point in the sequence $x, \phi(x), \phi^2(x), \dots.$ $v(x)$, of course, is idempotent – the values of $v(x)$ do not change when it is repeatedly composed with itself after the first time. For each periodic $x \in E$, $v^{-1}\{x\}$, the fibers of v^{-1} at x, forms a rooted tree with root x. It is easy to go the opposite direction as well, to form an endofunction from a permutation of rooted trees.

Proposition 3.9*. Vertebrates, or doubly-rooted trees, are linear orders of rooted trees.*

Proof. Call the first root of a vertebrate p_1 and the second root p_2, and the path along them a *spine*, consisting of vertebrae. Then each vertebrate has a rooted tree hanging off of it. Then, a vertebrate on E comes with a partition $E = E_1 + E_2 + \dots$ where each partition has a rooted tree, so that

$$V = A + A^2 + A^3 + \dots = \frac{1}{1 - A}.$$

Theorem 3.10 (Cayley's Theorem)*. The number of trees on $[n]$, a_n, is n^{n-2}.*

Proof. With species, we can prove this in just a few sentences. The number of doubly-rooted trees on $[n]$ is $n^2 a_n$. As we saw in the past two propositions, these vertebrates are linear assemblies of rooted trees, and endofunctions (n^n of them on $[n]$) are permutations of rooted trees. Because the number of linear orders coincides with the number of permutations, we have $n^2 a_n = n^n$. Tada!

4. Species Proof of Langrange Inversion Theorem

With these foundations and examples, we come to the main result of the paper: a species proof of the Lagrange Inversion Theorem, which was given in Joyal's original paper on combinatorial species. The Lagrange Inversion theorem arises from analysis, and has several applications in generating functions (say, in extracting the coefficients of A in the species $A = X \cdot \exp(A)$). The theorem is stated as

Theorem 4.1 (Lagrange Inversion Formula)*. For $v(x) = xR(v(x))$, we have $[x^n]g(v(x)) = \frac{1}{n}[t^{n-1}]g'(t)R(t)^n$.*

In this paper, we will use combinatorial species to prove a slightly differently stated version.

Theorem 4.2 (Lagrange Inversion Formula)*. Let R and F be species, and A_R be the species of r-enriched rooted trees. For $n \geq 1$, we have:*

$$F(A_R)[n] \equiv F' R^n [n-1]$$

where $\equiv$ denotes equipotence.

Proof. Firstly, note as covered earlier in Section 3, that

$$A_R = X \cdot R(A_R).$$

Taking the derivative, we have that

$$\begin{aligned} A'_R &= R(A_R) + XR'(A_R)A'_R \\ &= R(A_R) + C_R A'_R \end{aligned}$$

where $C_R = X \cdot R'(A_R)$, and by iteration,

$$\begin{aligned} A'_R &= R(A_R)(1 + C_R + C_R^2 + C_R^3 + \ldots) \\ &= R(A_R)\left(\frac{1}{1 - C_R}\right). \end{aligned}$$

Here, we can replace $\frac{1}{1-C_R}$ with S_R, the species of permutations of R. The next lemma provides a combinatorial interpretation of this.

Lemma 4.3. *The species $C_R = R'(A_R)$ is equivalent to the species of r-enriched contractions.*

Proof. A contraction is essentially an endofunction that is eventually constant, i.e. a function $\phi : E \to E$ such that there exists an $x_0 \in E$ where for large-enough n, $\phi^{(n)}(x) = x_0$ for $x \in E$.

Say we have an r-enriched contraction $\phi : E \to E$, which eventually converges to some $x_0 \in E$. For $E \setminus \{x_0\}$, we have the structure of an R'-assembly of A_R-structures, or a partitioning of A_R structures where each component is equipped with an R'-structure, which corresponds to $C_R = R'(A_R)$. □ We also need the following lemma.

Lemma 4.4. *We have $S(C_R) = D_R$, where S is the species of permutations and D_R is the species of r-enriched permutations.*

Proof. We can replace the rooted trees in Proposition 3.8, which states that $D = S(A)$, with contractions. At each point, we can see that by comparing the fibers of an endofunction with the fibers of a contraction, we realize that they are in bijection, and transfer the R-structures accordingly to form a bijection between the species of R-enriched endofunctions and R-enriched contractions. □

Next, we take the derivative of $F(A_R)$:

$$\begin{aligned} F(A_R)' &= F'(A_R)A'_R \\ &= F'(A_R)R(A_R)\frac{1}{1 - C_R} \\ &\equiv F'(A_R)R(A_R)D_R \end{aligned}$$

by Lemma 4.4.

Lemma 4.5 (Repartitioning lemma). *The objects of the species $G(A_R)D_R$, where G is a species, are partitions, $E = E_1 + E_2$, where E_1 is equipped with a G-structure γ and E_2 is equipped with an R-enriched function $\lambda : E_2 \to E$.*

Proof. From our definitions of species multiplication, we already know that the elements of $G(A_R)D_R[E]$ are a partition $E = F_1 + F_2$ equipped with some structures. On F_1, we have a forest of rooted trees equipped with a G-structure, and on F_2, we have an endofunction. Take the set of the roots of the trees in F_1, and call them E_1; denote its complement as E_2. The set E_2 plus the endofunction on F_2 defines an R-enriched function $\lambda : E_2 \to E$, whereas E_1 is equipped with a G-structure. In the other direction, we recover F_1 as the preimage of E_1 under λ, and F_2 as its complement. The proof can be better understood with this helpful figure from Joyal's paper:

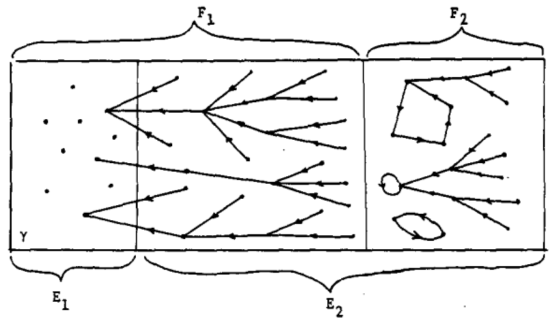

Figure 3. Joyal's paper says to "meditate" on this figure.

With this lemma, we can obtain the cardinality of $(G(A_R)D_R)[n]$. $E = [n]$, so $\lambda : E_2 \to E$ is exactly $\lambda : E_2 \to [n]$. Then, $(G(A_R)D_R)[n] = (G \cdot R^n)[n]$.

Applying this to $F(A_R)$, we have

$$\begin{aligned} F\left(A_R\right)[n] &\equiv F'\left(A_R\right) R\left(A_R\right) D_R[n-1] \\ &\equiv \left(F'R\right) R^{n-1}[n-1] \\ &= F'R^n[n-1] \end{aligned}$$

completing the proof. □

References

[1] Trevor Hyde. *Combinatorial Species and Generating Functions*. URL: `https://math.uchicago.edu/~tghyde/Hyde%20--%20Combinatorial%20species%20and%20generating%20functions.pdf`.

[2] André Joyal. "Une théorie combinatoire des séries formelles". In: *Advances in Mathematics* 42.1 (1981), pp. 1–82. ISSN: 0001-8708. DOI: `https://doi.org/10.1016/0001-8708(81)90052-9`. URL: `https://www.sciencedirect.com/science/article/pii/0001870881900529`.

[3] Brent A Yorgey. *A Combinatorial Theory of Formal Series (translation and commentary of Joyal's 1981 paper)*. 2018. URL: `http://ozark.hendrix.edu/~yorgey/pub/series-formelles.pdf`.