Herbert Wilf's Snake Oil Method

Using generating functions to evaluate combinatorial sums.

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Abstract

In this expository paper we will introduce *the Snake Oil method* that uses generating functions to deal with the evaluation of combinatorial sums. This method is capable of handling a great variety of sums involving binomial coefficients, and, within its limitations, on sums involving other combinatorial numbers. The philosophy is roughly this: don't try to evaluate the sum that you're looking at. Instead, find the generating function for the whole parameterized family of them, then read off the coefficients.

1 Motivation

Mathematicians, particularly those working on discrete mathematics, frequently have to evaluate or simplify complicated looking sums that involve combinatorial numbers. These sums turn up regularly in connection with problems in graphs, algorithms, enumeration, etc. (and they're fun, too!). It is therefore a very useful skill to be able to have different tools for tackling these kind of sums. In the past, one had to have built up a certain arsenal of special devices, the more the better, in order to be able to trot out the correct one for the correct occasion. Recently, however, a good deal of quite dramatic systematization has taken place, and there are unified methods for handling vast categories of the sums referred to above. The Snake Oil method is one such.

2 Introduction

The Snake Oil method is generally used when you have a sum over one variable and another free variable that is not summed over. In the example below, this free variable is n. The first step in the Snake Oil method is to write a generating function for the sum that sums through possible values of n. Then, this generating function can be expressed as a double sum, so we can swtich the order of the summation. Then, we will evaluate both of the summations from the inside to the outside. This will yield the generating function for the sum, and now we can determine the general form of a coefficient in the generating function to find the value of the sum at a particular value of n.

All of the steps above will become clearer in the example below.

Example 2.1. Find a closed form (if one exists) of the sum below

$$\sum_{k\geq 0}\binom{k}{n-k} \tag{1}$$

We notice that n is our free variable here, so let $a_n = \sum_{k\geq 0}\binom{k}{n-k}$. We can form a generating function for this sum that has the values for this sum at each value of n. Let $A(z)$ be the OGF corresponding to the sequence $\{a_n\}_{n\geq 0}$. We get

$$A(z) = \sum_n a_n z^n = \sum_{n\geq 0}\sum_{k\geq 0}\binom{k}{n-k}z^n$$

Now, we can switch the order of the double summation.

$$A(z) = \sum_{k\geq 0}\sum_{n\geq 0}\binom{k}{n-k}z^n$$

We would like to 'do' the inner sum, the one over n. The trick is to get the exponent of x to be exactly the same as the index that appears in the binomial coefficient. In this example the exponent of x is n, and n is involved in the downstairs part of the binomial coefficient in the form $n - k$. To make those the same, the correct medicine is to multiply inside the sum by $x - k$ and outside the inner sum by x_k, to compensate. The result is

$$\begin{aligned} A(z) &= \sum_{k\geq 0} z^k \sum_{n\geq 0}\binom{k}{n-k}z^{n-k} \\ &= \sum_{k\geq 0} z^k \sum_{r\geq 0}\binom{k}{r}z^r \\ &= \sum_{k\geq 0} z^k(1+z)^k \\ &= \sum_{k\geq 0}(z+z^2)^k \end{aligned}$$

So we have a geometric series with common ratio $z + z^2$. Thus

$$A(z) = \frac{1}{1-z-z^2}$$

It follows that

$$a_n = \sum_{k\geq 0}\binom{k}{n-k} = F_n$$

where the F_n's are the Fibonacci numbers.

3 What is the Snake Oil method, really?

The basic idea is what we might call the *external* approach to identities rather than the usual *internal* method.

To explain the difference between these two points of view, suppose we want to prove some identity that involves binomial coefficients. Typically such a thing would assert that some fairly intimidating-looking sum is in fact equal to such-and-such a simple function of n.

One approach that is now customary, thanks to the skillful exposition and deft handling by Knuth in [3], and by Graham, Knuth and Patashnik in [4], consists primarily of looking inside the summation sign (‘*internally*’), and using binomial coefficient identities or other manipulations of indices *inside* the summations to bring the sum to manageable form.

The Snake Oil method is complementary to the internal approach. In the *external*, or generatingfunctionological, approach one begins by giving a quick glance at the expression that is inside the summation sign, just long enough to spot the ‘free variables,’ i.e., what it is that the sum depends on after the dummy variables have been summed over. Suppose that such a free variable is called n.

Then instead of trying to grapple with the sum, just sweep it all under the rug, as follows:

The Snake Oil Method for Doing Combinatorial Sums

(a) Identify the free variable, say n, that the sum depends on. Give a name to the sum that we are working on; call it a_n.
(b) Let $A(z)$ be the OGF associated with the sequence a_0, a_1, a_2, …, i.e., $[z^n]A(z) = a_n$.
(c) Multiply the sum by z^n, and sum on n. Our generating function is now expressed as a double sum over n, and over whatever variable was first used as a dummy summation variable.
(d) Interchange the order of the two summations that we are now looking at, and perform the inner one in simple closed form. For this purpose it will be helpful to have a catalogue of series whose sums are known, such as the list in the last section of this paper.
(e) Try to identify the coefficients of the generating function of the answer, because those coefficients are what we want to find.

The success of the method depends on favorable outcomes of steps (d) and (e). What is surprising is the high success rate. It also has the ‘advantage’ of requiring hardly any thought at all; when it works, you know it, and when it doesn’t, that’s obvious too.

4 Examples

We will now discuss several examples to give the flavor and illustration of the Snake Oil method. The following series evaluations are most helpful in the examples that follow. First and foremost,

$$\sum_{r\geq 0}\binom{r}{k}x^r = \frac{x^k}{(1-x)^{k+1}} \qquad (k\geq 0). \tag{4.1}$$

Also useful are the binomial theorem

$$\sum_{r}\binom{n}{r}x^r = (1+x)^n \tag{4.2}$$

and the Catalan numbers generating function:

$$\sum_{n}\frac{1}{n+1}\binom{2n}{n}x^n = \frac{1}{2x}(1-\sqrt{1-4x}). \tag{4.3}$$

Example 4.1. Openers

Consider the sum

$$a_n = \sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} \qquad (m, n \geq 0)$$

Can it be that the same method will do this sum, without any further infusion of ingenuity? Indeed; just pour enough Snake Oil on it and it will be cured. Let $f(n)$ denote the sum in question, and let $F(x)$ be its opsgf. Dive in immediately by multiplying by x^n and summing over $n \geq 0$, to get

$$\begin{aligned}
F(x) &= \sum_{n\geq 0} x^n \sum_k \binom{n+k}{m+2k}\binom{2k}{k}\frac{(-1)^k}{k+1} \\
&= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^{-k} \sum_{n\geq 0}\binom{n+k}{m+2k} x^{n+k} \\
&= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^{-k} \sum_{r\geq k}\binom{r}{m+2k} x^{r} \\
&= \sum_k \binom{2k}{k}\frac{(-1)^k}{k+1} x^{-k} \frac{x^{m+2k}}{(1-x)^{m+2k+1}} \\
&= \frac{x^m}{(1-x)^{m+1}} \sum_k \binom{2k}{k}\frac{1}{k+1}\left\{\frac{-x}{(1-x)^2}\right\}^k \\
&= \frac{-x^{m-1}}{2(1-x)^{m-1}}\left\{1-\sqrt{1+\frac{4x}{(1-x)^2}}\right\} \\
&= \frac{-x^{m-1}}{2(1-x)^{m-1}}\left\{1-\frac{1+x}{1-x}\right\} \\
&= \frac{x^m}{(1-x)^m}.
\end{aligned}$$

The original sum is now unmasked: it is the coefficient of x^n in the last member above. But that is $\binom{n-1}{m-1}$, and we have our answer.

If the train of manipulations seemed long, consider that at least it's always the *same* train of manipulations, whenever the method is used, and also that with some effort a computer could be trained to do it!

Example 4.2. A discovery

Is it possible to write the sum

$$a_n = \sum_{k\leq n/2} (-1)^k \binom{n-k}{k} y^{n-2k} \qquad (n \geq 0) \tag{4.4}$$

in a simpler closed form?

This example shows the whole machine at work again, along with a few new wrinkles. The first step is to let F be the generating function for the sequence $\{f_n\}$, and try to find F instead of the sequence $\{f_n\}$.

To do that we multiply (4.4) on both sides by x^n and sum over $n \geq 0$ to obtain

$$F(x) = \sum_{n\geq 0} x^n \sum_{k\leq \frac{n}{2}} (-1)^k \binom{n-k}{k} y^{n-2k}.$$

The next step is invariably to interchange the summations and hope. To try to make the innermost summation as clean looking as possible, be sure to take to the outer sum any factors that depend only on k. This yields

$$F(x) = \sum_{k} (-1)^k y^{-2k} \sum_{n\geq 2k} \binom{n-k}{k} x^n y^n.$$

Now focus on (4.4), and try to make the inner sum look like that. If in our inner sum the powers of x and y were $x^{n-k}y^{n-k}$, then those exponents would match exactly the upper story of the binomial coefficient $\binom{n-k}{k}$, and so after a change of dummy variable of summation we would be looking exactly at the left side of (4.4)

Hence we next multiply inside the inner sum by $x^{-k}y^{-k}$, and outside the inner sum by $x^k y^k$. Now we have

$$\begin{aligned}
F(x) &= \sum_{k} (-1)^k y^{-2k} x^k y^k \sum_{n\geq 2k} \binom{n-k}{k} x^{n-k} y^{n-k} \\
&= \sum_{k} (-1)^k x^k y^{-k} \sum_{a\geq k} \binom{a}{k} (xy)^a \\
&= \sum_{k\geq 0} (-1)^k x^k y^{-k} \frac{(xy)^k}{(1-xy)^{k+1}} \\
&= \frac{1}{1-xy} \sum_{k\geq 0} \left\{ \frac{-x^2}{1-xy} \right\}^k \\
&= \frac{1}{1-xy} \frac{1}{1+\frac{x^2}{1-xy}} \\
&= \frac{1}{1-xy+x^2}.
\end{aligned}$$

We now expand in partial fractions to obtain a closed form for the sum. This gives

$$\begin{aligned}
F(x) &= \frac{1}{(1-x_+)(1-xx_-)} \\
&= \frac{x_+}{(x_+-x_-)(1-xx_+)} - \frac{x_-}{(x_+-x_-)(1-xx_-)}
\end{aligned}$$

where

$$x_\pm = \frac{1}{2}(y \pm \sqrt{y^2-4}).$$

Hence, for $n \geq 0$ the coefficient of x^n is

$$f_n = \frac{1}{\sqrt{y^2-4}} \left\{ \left(\frac{y+\sqrt{y^2-4}}{2} \right)^{n+1} - \left(\frac{y-\sqrt{y^2-4}}{2} \right)^{n+1} \right\}.$$

We now have our answer, but just to demonstrate the effectiveness of cleanup operations, let's invest a little more time in making the answer look nicer. Because of the ubiquitous appearance of $\sqrt{y^2-4}$ in the answer, we replace y formally by $x+(1/x)$. Then

$$\sqrt{y^2-4} = x - \frac{1}{x},$$

and our formula becomes

$$\sum_{k\le\frac{n}{2}} (-1)^k \binom{n-k}{k} (x^2+1)^{n-2k} x^{2k} = \frac{x^{2(n+1)}-1}{x^2-1} \qquad (n \ge 0).$$

Finally we write $t = x^2$ to obtain the pretty evaluation

$$\sum_{k\le\frac{n}{2}} (-1)^k \binom{n-k}{k} (t+1)^{n-2k} t^k = \frac{1-t^{n+1}}{1-t} \qquad (n \ge 0). \tag{4.5}$$

For instance, the value $t = 1$ gives

$$\sum_{k\le\frac{n}{2}} (-1)^k \binom{n-k}{k} 2^{n-2k} = n+1 \qquad (n \ge 0).$$

As a final touch, we can read off the coefficient of t^m in (4.5) to discover the interesting fact

$$\sum_{k\le\frac{n}{2}} (-1)^k \binom{n-k}{k} \binom{n-2k}{m-k} = \begin{cases} 1, & \text{if } 0 \le m \le n; \\ 0, & \text{otherwise.} \end{cases}$$

Example 4.3.

Evaluate the sums

$$a_n = \sum_k \binom{n+k}{2k} 2^{n-k} \qquad (n \ge 0) \tag{4.6}$$

Without stopping to think, let F be the opsgf of the sequence, multiply both sides of (4.6) by x^n, sum over $n \geq 0$, and interchange the two sums on the right. This produces

$$\begin{aligned}
F &= \sum_k 2^{-k} \sum_{n\geq 0} \binom{n+k}{2k} 2^n x^n \\
&= \sum_k 2^{-k}(2x)^{-k} \sum_{n\geq 0} \binom{n+k}{2k} (2x)^{n+k} \\
&= \sum_{k\geq 0} 2^{-k}(2x)^{-k} \frac{(2x)^{2k}}{(1-2x)^{2k+1}} \\
&= \frac{1}{1-2x} \sum_{k\geq 0} \left\{ \frac{x}{(1-x)^2} \right\}^k \\
&= \frac{1}{1-2x} \frac{1}{1-\frac{x}{(1-2x)^2}} \\
&= \frac{1-2x}{(1-4x)(1-x)} \\
&= \frac{1}{3}\left(\frac{2}{1-4x} + \frac{1}{1-x} \right)
\end{aligned}$$

It is now trivial to read off the coefficient of x^n on both sides and discover the answer:

$$\sum_k \binom{n+k}{2k} 2^{n-k} = \frac{1}{3}(2^{2n+1}+1) \qquad (n \geq 0).$$

Example 4.4.

The next example is a sum that we won't succeed in evaluating in a neat, closed form. However, the generating function that we obtain will be rather tidy, and that is about the most that can be expected from this family of sums.

The sum is

$$f_n(y) = \sum_k \binom{n}{k} \binom{2k}{k} y^k \qquad (n \geq 0). \tag{4.7}$$

We follow the usual prescription. Define $F(x,y) = \sum_{n\geq 0} f_n(y)x^n$. To find F, multiply (4.7) by x^n, sum over $n \geq 0$ and interchange the inner and outer sums, to obtain

$$\begin{aligned}
F(x,y) &= \sum_k \binom{2k}{k} y^k \sum_{n\geq 0} \binom{n}{k} x^n \\
&= \sum_k \binom{2k}{k} y^k \frac{x^k}{(1-x)^{k+1}} \\
&= \frac{1}{1-x} \sum_k \binom{2k}{k} \left(\frac{xy}{1-x} \right)^k .
\end{aligned}$$

Now since

$$\sum_k \binom{2k}{k} z^k = \frac{1}{\sqrt{1-4z}},$$

we obtain

$$\begin{aligned} F(x,y) &= \frac{1}{(1-x)\sqrt{1-\frac{4xy}{1-x}}} \\ &= \frac{1}{\sqrt{1-x}\sqrt{1-x(1+4y)}}. \end{aligned}$$

For general values of y, that's about all we can expect. There are two special values of y for which we can go further. If $y = -1/4$, we find that

$$\sum_k \binom{2k}{k}\binom{n}{k}\left(-\frac{1}{4}\right)^k = 2^{-2n}\binom{2n}{n} \qquad (n \geq 0).$$

If $y = -1/2$, then

$$\begin{aligned} F(x,-1/2) &= 1/\sqrt{1-x^2} \\ &= \sum_m \binom{2m}{m}\left(\frac{x}{2}\right)^{2m}. \end{aligned}$$

Hence we have Reed Dawson's identity

$$\sum_k \binom{2k}{k}\binom{n}{k}(-1)^k 2^{-k} = \begin{cases} \binom{n}{n/2}2^{-n} & \text{if } n \geq 0 \text{ is even,} \\ 0 & \text{if } n \geq 0 \text{ is odd,} \end{cases}$$

and Snake Oil triumphs again.

Example 4.5.

Suppose we have two complicated sums and we want to show that they're the same. Then the generating function method, if it works, should be very easy to carry out. Indeed, one might just find the generating functions of each of the two sums independently and observe that they are the same.

Suppose we want to prove that

$$\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k}2^k \qquad (m, n \geq 0)$$

without evaluating either of the two sums. Multiply on the left by x_n, sum on $n \geq 0$ and interchange the summations, to arrive at

$$\begin{aligned} \sum_k \binom{m}{k} x^{-k} \sum_{n\geq 0}\binom{n+k}{m} x^{n+k} &= \sum_k \binom{m}{k} x^{-k} \cdot \frac{x^m}{(1-x)^{m+1}} \\ &= \frac{x^m}{(1-x)^{m+1}}\left(1+\frac{1}{x}\right)^m \\ &= \frac{(1+x)^m}{(1-x)^{m+1}}. \end{aligned}$$

If we multiply on the right by x^n, etc., we find

$$\begin{aligned}\sum_k \binom{m}{k} 2^k \sum_{n\geq 0} \binom{n}{k} x^n &= \frac{1}{1-x} \sum_k \binom{m}{k} \left(\frac{2x}{1-x}\right)^k \\ &= \frac{1}{1-x}\left(1+\frac{2x}{1-x}\right)^m \\ &= \frac{(1+x)^m}{(1-x)^{m+1}}.\end{aligned}$$

Hence the two sums are equal, even if we don't know what they are!

The followng problem is slightly harder because the standard idea of Snake Oil doesn't quite lead to a solution.

Example 4.6. (Moriati). For given n and p evaluate

$$\sum_k \binom{2n+1}{2p+2k+1}\binom{p+k}{k} \tag{4.8}$$

In order to have shorter formulas let us introduce $r = p + k$. If we assume that n is the free variable then the required sum is equal to

$$f(n) = \sum_r \binom{2n+1}{2r+1}\binom{r}{k}.$$

Take $F(x) = \sum_n x^{2n+1} f(n)$. This is somehow natural since the binomial coefficient contains the term $2n+1$. Now we have

$$\begin{aligned}F(x) &= \sum_n x^{2n+1} \sum_r \binom{2n+1}{2r+1}\binom{r}{p} \\ &= \sum_r \binom{r}{p} \sum_n \binom{2n+1}{2r+1} x^{2n+1}.\end{aligned}$$

Since

$$\sum_n \binom{2n+1}{2r+1} x^{2n+1} = \frac{1}{2} x^{2r+1}\left\{\frac{1}{(1-x)^{2r+2}} + \frac{1}{(1+x)^{2r+2}}\right\},$$

we get

$$\begin{aligned}F(x) &= \frac{1}{2}\cdot\frac{x}{(1-x)^2}\sum_r \binom{r}{p}\left[\frac{x^2}{(1-x)^2}\right]^r + \frac{1}{2}\cdot\frac{x}{(1+x)^2}\sum_r \binom{r}{p}\left[\frac{x^2}{(1+x)^2}\right]^r \\ &= \frac{1}{2}\cdot\frac{x^{2p+1}}{(1-2x)^{p+1}} + \frac{1}{2}\cdot\frac{x^{2p+1}}{(1+2x)^{p+1}} \\ &= \frac{x^{2p+1}}{2}\left[(1+2x)^{-(p+1)} + (1-2x)^{-(p+1)}\right],\end{aligned}$$

implying

$$f(n) = \frac{1}{2}\left[\binom{-p-1}{2n-2p} 2^{2(n-p)} + \binom{-p-1}{2n-2p} 2^{2(n-p)}\right],$$

and after simplification

$$f(n) = \binom{2n-p}{2n-2p} 2^{2(n-p)}.$$

5 Snake Oil vs. hypergeometric functions

Many combinatorial identities are special cases of identities in the theory of hypergeometric series (we'll explain that remark, briefly, in a moment). However, the Snake Oil method can cheerfully deal with all sorts of identities that are not basically about hypergeometric functions. So the approaches are complementary.

A hypergeometric series is a series

$$\sum_k T_k$$

in which the ratio of every two consecutive terms is a rational function of the summation variable k. That means that

$$\frac{T_{k+1}}{T_k} = \frac{P(k)}{Q(k)},$$

where P and Q are polynomials, and it takes in a lot of territory. Many binomial coefficient identities, including all of the examples in this chapter so far, are of this type. There are some general tools for dealing with such sums, and these are very important considering how frequently they occur in practice.

In this example we want to emphasize that the scope of the Snake Oil method includes a lot of sums that are not hypergeometric. Consider, for instance, the following sum

$$f(n) = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} B_k,$$

where the $[\ ]$ are the Stirling numbers of the first kind, and the B's are the Bernoulli numbers.

Now one thing, at least, is clear from looking at this sum: it is not hypergeometric. The ratio of two consecutive terms is certainly not a rational function of k. The Snake Oil method is, however, unfazed by this turn of events. If you follow the method exactly as before, you could define $F(x)$ to be the egf of the sequence $\{f(n)\}$, multiply by $x^n/n!$, sum on n, interchange the indices, etc., and obtain

$$\begin{aligned} F(x) &= \sum_n \frac{f(n)x^n}{n!} \\ &= \sum_n \frac{x^n}{n!} \sum_k \begin{bmatrix} n \\ k \end{bmatrix} B_k \\ &= \sum_k B_k \sum_n \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!} \\ &= \sum_k B_k \left\{ \frac{1}{k!} \left(\log \frac{1}{1-x} \right)^k \right\}, \end{aligned}$$

and making the change of variables $u = \log \frac{1}{1-x}$, we get

$$\begin{aligned} &= \sum_k \frac{B_k}{k!} u^k \\ &= \frac{u}{e^u - 1} \\ &= \frac{1-x}{x} \log \frac{1}{1-x}. \end{aligned}$$

If we now read off the coefficient of $x^n/n!$ on both sides, we find that the unknown sum is

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} B_k = -\frac{(n-1)!}{n+1} \qquad (n \geq 1) \tag{5.1}$$

6 The scope of the Snake Oil method

The success of the Snake Oil method depends upon being given a sum to evaluate in which there is a free variable that appears in only one place. Then, after interchanging the order of the summations, one finds one of the basic power series (4.1) or (4.2) to sum.

At the risk of diminishing the charm of the method somewhat by adding gimmicks to it, one must remark that in many important cases this limitation on the scope is easy to overcome. This is because it frequently happens that when an identity is presented that has a free variable repeated several times, that identity turns out to be a special case of a more general identity in which each of the repeated appearances of the free variable is replaced by a different free variable. Before abandoning the method on some given problem, this possibility should be explored.

Consider the identity

$$\sum_i \binom{n}{i} \binom{2n}{n-i} = \binom{3n}{n}$$

At first glance the possibilities for successful Snake Oil therapy seem dim because of the multiple appearances of n in the summand. However, if we generalize the identity by splitting the appearances of n into different free variables, we might be led to consider the sum

$$\sum_i \binom{n}{i} \binom{m}{r-i}$$

which is readily evaluated by the Snake Oil method. It is characteristic of the subject of identities that it is usually harder to prove special cases than general theorems. Multiple appearances of a free variable are often a hint that one should try to find a suitable generalization.

7 WZ pairs prove harder identities

Computers can now find proofs of combinatorial identities, including most of the identities that we did by the Snake Oil method in the previous section, as well as many, many more.

This doesn't mean that it was a waste of time to have learned the Snake Oil method. There were identities that Snake Oil handled that the method of [WZ] (which is the basis of one of the prime computer algorithms) cannot deal with, like the fact that $\sum_{k\geq 0}\binom{k}{n-k} = F_n$ for $n \geq 0$. Another one that offers no hope to the WZ method is (5.1), which involves Stirling and Bernoulli numbers. Also, to use the Snake Oil method, one doesn't need to know the right hand side of the identity in advance; the method will find it. The methods that we are describing will prove a given identity, but it won't discover the identity for itself.

With those disclaimers, however, it is fair to say that the WZ method is quite versatile, and seems able to handle in a unified way some of the knottiest identities known.

References

[1] Flajolet, Philippe, and Robert Sedgewick. 2013. *Analytic Combinatorics.* Cambridge University Press.

[2] Wilf, Herbert S. 1994. *Generatingfunctionology.* Academic Press, Boston.

[3] Knuth, Donald E. 1968-73. *The Art of Computer Programming, vol. 1: Fundamental Algorithms.* (1968); *vol. 2: Seminumerical Algorithms.* (1969); *vol. 3: Sorting and Searching.* (1973). Addison-Wesley, Reading, MA.

[4] Graham, Ronald L., Knuth, Donald. E., and Patashnik, Oren. 1989. *Concrete Mathematics.* Addison-Wesley, Reading, MA.

[5] Rubinstein-Salzedo, S. 2021. *Generating Funtions.* Lecture notes for Euler Circle, Fall 2021.

Appendix

Some useful power series

Generatingfunctionologists need reference lists of known power series and other series that occur frequently in applications of the theory. Here is such a list.

$$\frac{1}{1-x} = \sum_{n\geq 0} x^n \tag{7.1}$$

$$\log\frac{1}{1-x} = \sum_{n\geq 1} \frac{x^n}{n} \tag{7.2}$$

$$e^x = \sum_{n\geq 0} \frac{x^n}{n!} \tag{7.3}$$

$$\sin x = \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{7.4}$$

$$\cos x = \sum_{n\geq 0} (-1)^n \frac{x^{2n}}{(2n)!} \tag{7.5}$$

$$(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k \tag{7.6}$$

$$\frac{1}{(1-x)^{k+1}} = \sum_{n} \binom{n+k}{n} x^n \tag{7.7}$$

$$\tan^{-1} x = \sum_{n\geq 0} (-1)^n \frac{x^{2n+1}}{2n+1} \tag{7.8}$$

$$\frac{1}{\sqrt{1-4x}} = \sum_{k} \binom{2k}{k} x^k \tag{7.9}$$

$$\frac{1}{2x}\left(1-\sqrt{1-4x}\right) = \sum_{n} \frac{1}{n+1}\binom{2n}{n} x^n \tag{7.10}$$