POLYA ENUMERATION THEOREM

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ABSTRACT. The Pólya Enumeration Theorem is a powerful tool in combinatorics. In this paper, we will build up to the theorem (along with its proof) and some applications.

1. Group Theory

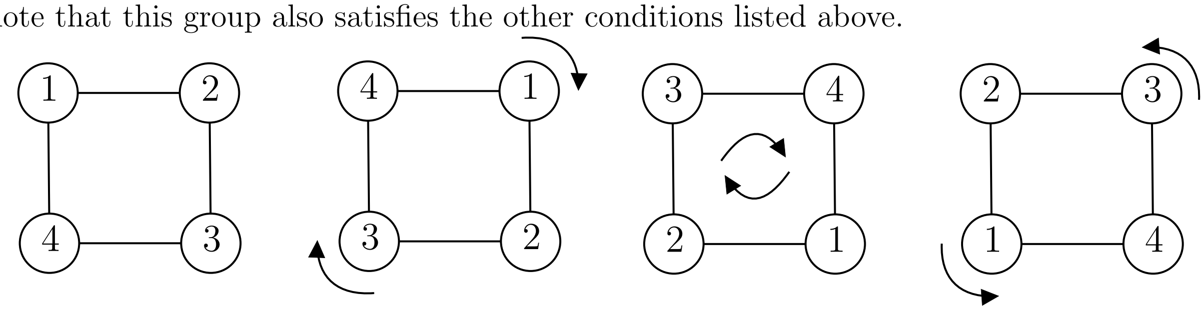
We assume basic knowledge of group theory.

Definition 1.1. Recall quickly that a group is a set G along with some operation $\cdot$ such that the following 4 properties are satisfied:

(1) It is closed: for all $g, h \in G$, $g \cdot h \in G$.
(2) It is associative: for all $a, b, c \in G$, $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
(3) It has an identity: there exists an element $e \in G$ such that for all $g \in G$, $e \cdot g = g \cdot e = g$.
(4) All elements have inverses: for each element $g \in G$, there exists an element $g^{-1} \in G$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$.

(Sometimes, we leave out the $\cdot$ when writing $a \cdot b$, making it just ab.)

Example. The rotations of a square is a group called C_4, consisting of 4 elements that are the rotations (clockwise) of 0° (identity), 90°, 180°, and 270°, named e, a, b, c respectively. The operation $\cdot$ composes the rotations together. Note that e is the identity element, and $a \cdot c = c \cdot a = e$, so a and c are inverses of each other. $b \cdot b = e$, so b is its own inverse. Quickly note that this group also satisfies the other conditions listed above.



Identity, 0° $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ Rotation by 90° $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$ Rotation by 180° $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$ Rotation by 270° $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$

For the purposes of Pólya's Enumeration Theorem, it is often useful to think of a group element as a permutation. For example, if we enumerate the vertices of a square, we can turn the C_4 group into permutations. The rotation of 0° from above is the permutation 1234, since every corner goes to itself. The rotation of 90° (cw) maps 1 to where 2 was, 2 to where

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3 was, 3 to where 4 was, and 4 to where 1 was, so in matrix notation, it is the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} = 2341$$

We can consider these permutations as functions on a set. For example, the C_4 group from above can be considered as functions on the set $X = \{1, 2, 3, 4\}$, representing the labelled vertices of the square.[1] If the rotation of 90° (cw) is the function σ, then $\sigma(1) = 2, \sigma(2) = 3, \sigma(3) = 4$, and $\sigma(4) = 1$.

When a group acts on a set like this, we call it a **group action**, denoted $G \curvearrowright X$. If $g \in G$ and $x \in X$, we also use the notation $g \cdot x$ to denote the result when element g acts on x. For example, if g is again the rotation of 90° (cw) and x is the vertex labelled 1, then $g \cdot x$ is the vertex labelled 2.

More formally, we can define a group action by imposing certain restrictions.

Definition 1.2. A group action of a group G acting on a set X, or $G \curvearrowright X$, must satisfy

(1) $e_G \cdot x = x$ for all $x \in X$, where e_G is the identity element of G.
(2) It is associative: $g \cdot (h \cdot x) = (gh) \cdot x$.

Now, we can define some terminology:

Definition 1.3. Let G be a group action on a set X, denoted $G \curvearrowright X$.

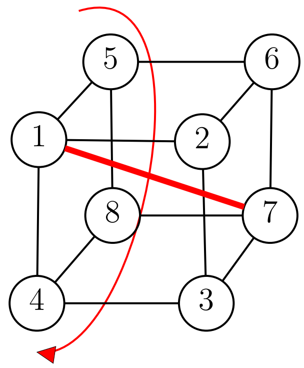
- Given some $x \in X$, the **orbit** of x (denoted $\mathrm{Orb}_G(x)$) is the set of all elements $y \in X$ such that $gx = y$ for some $g \in G$. Informally, $\mathrm{Orb}_G(x)$ is all elements that x can reach through G.
- Given some $x \in X$, the **stabilizer** of x (denoted $\mathrm{Stab}_G(x)$) is the set of all elements $g \in G$ such that $gx = x$. In other words, it is the set of all g that fix x.
- Given $g \in G$, the **fixed points** of g (denoted $\mathrm{Fix}(g)$) is the set of all $x \in X$ that have $gx = x$, or all elements $x \in X$ that are fixed by g.

Example. Let's again use the C_4 group acting on the set of vertices X. If $x \in X$ is the vertex labelled 1, then $\mathrm{Orb}_{C_4}(x)$ is just $X = \{1, 2, 3, 4\}$, because the 4 rotations each map it to a different vertex. Also, $\mathrm{Stab}_{C_4}(x)$ is just the identity element, because only the identity element keeps the vertex labelled 1 fixed.

This isn't all too interesting. Let's instead consider the group G of rotations of a cube, acting on the set of 8 vertices $X = \{1, 2, \ldots, 8\}$. Say $x \in X$ is the vertex labelled 1. The reader can verify that there are rotations that carry vertex 1 to every other vertex in the cube, thus $\mathrm{Orb}_G(x) = X$.

However, unlike before, there are actually 3 rotations (including the identity) that rotate around the diagonal from vertex 1 to vertex 7, depicted below. These rotations do not move vertex 1, thus $\mathrm{Stab}_G(x)$ has 3 elements.

[1] Note that for the same group of rotations, there actually are multiple different sets it could act on. For example, it could act on the set of squares whose 4 vertices are colored white or black, which we will soon see later.



$$|Stab_G(x)| = 3$$

What if we wanted to find the number of rotations of a cube? One way is to first perform one of the 3 rotations around the vertex 1 (the stabilizers of vertex 1), then rotate the vertex 1 to one of the 8 vertices in the cube (the orbit of vertex 1). This yields $3 \cdot 8 = 24$ rotations.

This inspires the following theorem:

Theorem 1.4 (Orbit-Stabilizer Theorem)**.** *Let $G \curvearrowright X$. For any $x \in X$,*

$$|Orb_G(x)| \cdot |Stab_G(x)| = |G|$$

.

Proof. We consider a relatively informal proof here. Let $G \curvearrowright X$, and say the elements of G are $g_1, g_2, \dots$ and the elements of X are $y_1, y_2, \dots$. Then, choose any arbitrary $x \in X$. The following table has a checkmark at the cell at g_i and y_j if $g_i \cdot x = y_j$, or if g_i "maps" x to y:

	y_1	y_2	y_3	$\cdots$
g_1	✓			
g_2		✓		
g_3		✓		
$\vdots$				$\ddots$

Note that each row must have 1 and only 1 checkmark, since $g_i \cdot x$ is 1 and only 1 element in X. There are $|G|$ rows, so there must be $|G|$ total checkmarks.

Now, each column for some $y \in X$ counts the number of g_i such that g_i maps x to y, or the size of the set $S_{xy} = \{g \in G : g \cdot x = y\}$. Note that if y is not in the orbit of x, by definition, $|S_{xy}| = 0$.

However, if y is in the orbit of x, then we can actually prove that $|S_{xy}| = |\operatorname{Stab}_G(x)|$. First, pick any arbitrary $h \in S_{xy}$, which must exist since $y \in \operatorname{Orb}_G(x)$. Then, we establish a function $\phi : S_xy \to \operatorname{Stab}_G(x)$ such that $\phi(g) = h \cdot g$. However, this function has an inverse! Note that the function $\psi(g) = h^{-1}g$ inverts ϕ, so ϕ is a bijection. Thus, we have shown $|S_{xy}| = |\operatorname{Stab}_G(x)|$ for all $y \in \operatorname{Orb}_x(G)$.

Finally, we have that the total number of checkmarks is $|\operatorname{Orb}_x(G)| \cdot |\operatorname{Stab}_G(x)|$. However, the total number of checkmarks is also $|G|$, from before! We conclude that $\therefore$ $|\operatorname{Orb}_G(x)| \cdot |\operatorname{Stab}_G(x)| = |G|$. ∎

2. Burnside's Lemma

Definition 2.1. A **partition** of a set X is a collection of disjoint subsets of X, called cells.

Example. $\{\{1, 2, 3\}, \{4\}\}$ is a partition of $\{1, 2, 3, 4\}$.

Definition 2.2. An equivalence relation $\sim$ on a set X is a relation that satisfies:

(1) Reflectivity: $x \sim x$
(2) Symmetry: $x \sim y$ implies $y \sim x$
(3) Transitivity: $x \sim y$ and $y \sim z$ implies $x \sim z$

Proposition 2.3. *Given an equivalence relation $\sim$ on X, we can form a partition. The cell of an element a is given by*

$$\{x \in X : x \sim a\}$$

For now, we state this without proof. It is very useful for PET to consider the partition into orbits.

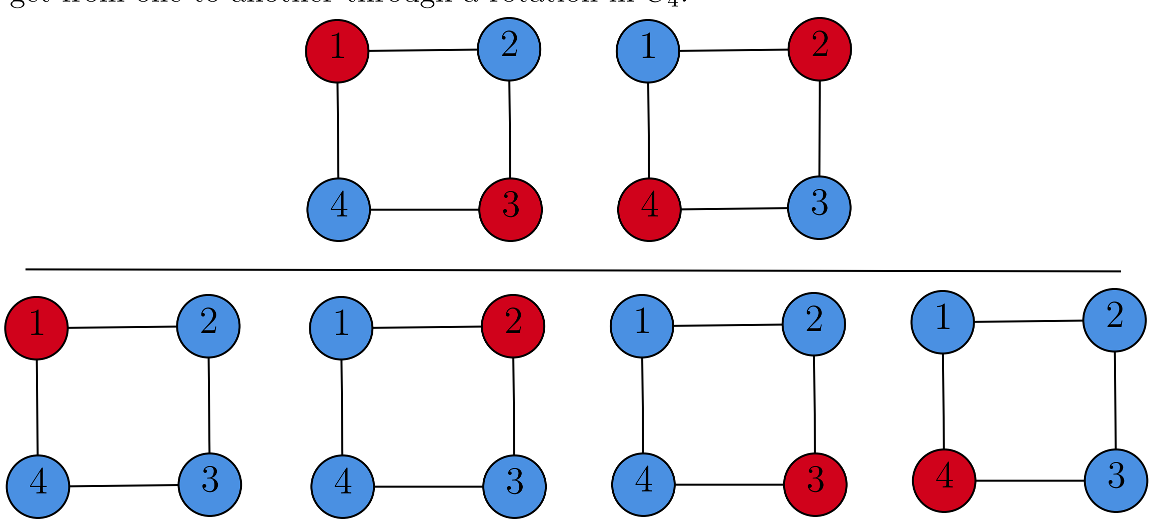
Proposition 2.4. *If $G \curvearrowright X$, the orbits of X are a partition of X. Let this partition be denoted X/G.*

Proof. We define an equivalence relation on the elements of X, such that $a \sim b$ iff a is in the orbit of b (and vice versa). In other words, $a \sim b$ iff there exists $g \in G$ such that $g \cdot a = b$.

This is an equivalence relation because it is reflexive ($a \sim a$ because $e \cdot a = a$), it is symmetric ($a \sim b$ implies $g \cdot a = b$, so $g^{-1} \cdot b = a$ and $b \sim a$), and it is transitive ($a \sim b$ and $b \sim c$ implies $ga = b, hb = c$, so $hga = hb = c$, and $a \sim c$). ∎

Example. Consider the set X of squares with labelled vertices, each colored either red or blue. Also, consider the group C_4 of the rotations of a square, from before, such that $C_4 \curvearrowright X$.

The following squares on each row are in the same orbit with respect to C_4, because you can get from one to another through a rotation in C_4.



There are $2^4 = 16$ ways to color the 4 vertices of the square. However, it begs the question: if you consider 2 colorings to be the same if you can get from one to another via a rotation, how many colorings are there? In other words, how many orbits are there up to C_4, or what is the size of X/C_4?

To answer these questions, we use Burnside's Lemma, a powerful theorem that can count the orbits of a set up to some group action.

Theorem 2.5 (Burnside's Lemma)**.** *Let $G \curvearrowright X$.*

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|\text{Fix}(g)|$$

Proof. First note that if $x \in \text{Fix}(g)$, then x is fixed by g and $g \in \text{Stab}_G(x)$. Thus,

$$\sum_{g\in G}|\text{Fix}(g)| = \sum_{x\in X}|\text{Stab}_G(x)|$$

Now, from the Orbit-Stabilizer Theorem, we have

$$|\text{Orb}_G(x)| \cdot |\text{Stab}_G(x)| = |G| \implies |\text{Stab}_G(x)| = \frac{|G|}{|\text{Orb}_G(x)|}$$

$$\sum_{x\in X}|\text{Stab}_G(x)| = \sum_{x\in X}\frac{|G|}{|\text{Orb}_G(x)|} = |G|\sum_{x\in X}\frac{1}{|\text{Orb}_G(x)|}$$

Consider each orbit: if an orbit has n elements, then each of those n elements will contribute $\frac{1}{|\text{Orb}_G(x)|} = \frac{1}{n}$ to the sum, so in total, each orbit will contribute 1 to the sum. Therefore, $\sum_{x\in X}\frac{1}{|\text{Orb}_G(x)|}$ is just counting the number of orbits!

$$\sum_{g\in G}|\text{Fix}(g)| = |G|\sum_{x\in X}\frac{1}{|\text{Orb}_G(x)|} = |G| \cdot |X/G|$$

$$|X/G| = \frac{1}{|G|}\sum_{g\in G}|\text{Fix}(g)|$$

■

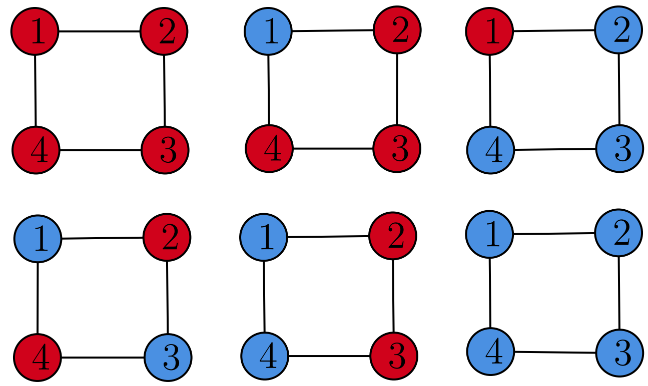
Example. Let's answer the question from before: how many ways are there to color the vertices of a square red or blue, given that two colorings are the same if one is a rotation of the other?

$$|X/C_4| = \frac{1}{|C_4|}\sum_{g\in G}|\,\text{Fix}(g)|$$

$|C_4| = 4$. The rotation of 0° fixes all 16 of the colorings. The rotations of 90° and 270° will only keep the coloring the same if all the vertices are the same color, so they only fix 2 colorings each. Finally, the rotation of 180° will fix colorings that have opposite vertices the same color, which makes for 4 total colorings.

$$|X/C_4| = \frac{1}{4}\cdot(16+2+4+2) = 6$$

There are 6 total colorings up to rotation, shown below.



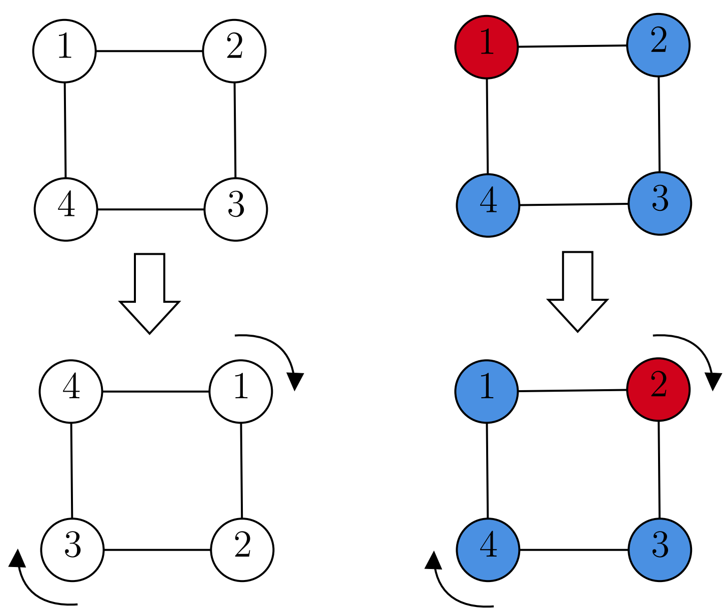
3. Colorings

Pólya's Enumeration Theorem deals with the colorings of a finite set up to a group action. If C is a finite set of colors and D is a finite set, let C^D denotes the possible colorings of D. Formally, C^D is the set of all functions $f : D \to C$.

Example. Our example of coloring the vertices of a square from before can be instead formulated as C^D, where $C = \{\text{Red}, \text{Blue}\}$. Then, $D = \{1, 2, 3, 4\}$, one label for each vertex.

Note that an element $g \in G$ can intuitively act on the set of colorings $f \in C^D$ as well; to define it formally, $g \star f(d) = f(g^{-1}(d))$.

Example. Say the rotation of $90°$ is the function $\sigma : D \to D$. Now, let f be the coloring shown on the top right, below, such that $f(1) = \text{Red}$ and $f(2) = f(3) = f(4) = \text{Blue}$. We can allow σ to act on f using our formal definition: say $\sigma \star f = f'$. Then, $f'(1) = f(\sigma^{-1}(1)) = f(4) = \text{Blue}$, whereas $f'(2) = f(\sigma^{-1}(2)) = f(4) = \text{Red}$, as shown in the coloring on the bottom right below.



Rotation by $90°$

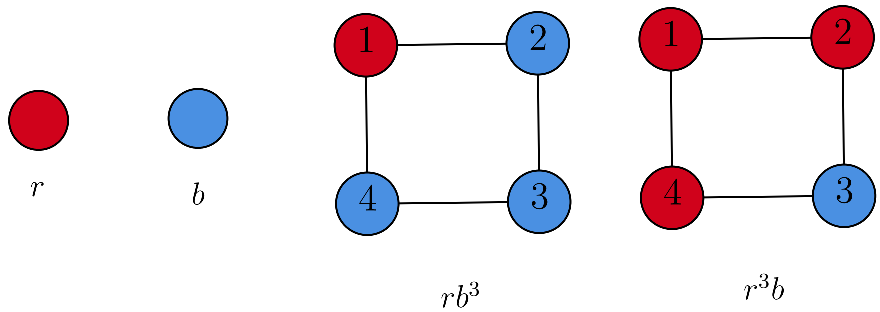
Definition 3.1. We can partition C^D into orbits. All orbits of C^D are C^D/G, and these orbits are called **configurations**.

Note that the configurations are actually what we're interested in: they're the colorings up to some group action G. We could find the number of configurations with Burnside's Lemma, but Pólya's Enumeration Theorem extends this with the idea of weights.

4. Weights and Generating Functions

Definition 4.1. We can assign **"weights"** to each color. For all $c \in C$, $w(c)$ denotes the weight of the color c.[2] Then, we can also define the weight of a coloring $f \in C^D$, which will be the product of the weights of the colors: $W(f) = \prod_{d \in D} w(f(d))$.

Example. Say that the weight of the color red is the variable r, and say the weight of the color blue is the variable b. The weights of some colorings are shown below.



Proposition 4.2. *If two colorings $f_1, f_2 \in C^D$ are in the same configuration, then they have the same weight.*

Proof. This should intuitively make sense; if you rotate or move a structure around, you never change the colors, so the total number of colors should stay the same.

To prove it rigorously, we consider two colorings $f_1, f_2 \in C^D$ that are in the same configuration, such that $g \star f_1 = f_2$.

$$W(f_2) = \prod_{d \in D} w(f_2(d)) = \prod_{d \in D} w(f_1(g^{-1}d))$$

However, $g^{-1}d$ for all D is just a permutation of D! Therefore, we can write it as

$$\therefore W(f_2) = \prod_{d \in D} w(f_1(d)) = W(f_1)$$

■

This allows us to define $W(f)$, where f is some configuration of C^D, as the weight of all the colorings in the configuration, which are the same.

Definition 4.3. Let F be the set of all configurations of C^D up to G. The **Configuration Generating Function (CGF)** is the multivariate generating function

$$\sum_{f \in F} W(f)$$

Note that the CGF tells us a great deal about the configurations, telling us exactly how many configurations there are with each combination of colors.

[2]Usually, $w(c)$ is treated as a variable in a generating function and so it never gets a real value.

Example. The configurations of the colorings of the vertices of a square in either red and blue up to rotation, from before, is

$$r^4 + r^3 b + 2r^2 b^2 + rb^3 + b^4$$

There is 1 coloring with 3 reds and 1 blue, but there are 2 colorings with 2 reds and 2 blues.

5. Cycle Index

Let's define the Cycle Index Polynomial, which is an algebraic way to store information about a group action. It's one of the primary tools for Pólya's Enumeration Theorem.

Definition 5.1. The **Cycle Index Polynomial** of a group action $G \curvearrowright C^D$ is the multivariate polynomial

$$Z_G(t_1, t_2, t_3, \ldots) = \frac{1}{|G|} \sum_{g \in G} t_1^{c_1(g)} t_2^{c_2(g)} t_3^{c_3(g)} \cdots$$

, where $c_i(g)$ is the number of cycles of length i in g (g is treated as a permutation of D).

Example. Consider the rotation of 90° in a square, which is the permutation 2341, or in cycle notation, is (1234). There is 1 cycle of length 4, so its term in the cycle index polynomial will be t_4.

The rotation of 180°, meanwhile, is the permutation 3412, which is (13)(24). There are 2 cycles of length 2, so its term in the cycle index polynomial will be t_2^2.

In total, the cycle index polynomial for the C_4 group on the vertices of a square is

$$\frac{1}{4}(t_1^4 + t_2^2 + 2t_4)$$

Also, it's important to note the relation between cycles and the number of fixed points. If a permutation fixes a coloring, then note that all elements within each cycle must be colored the same.

Example. In the rotation of 90°, all elements in the cycle (1234) must be the same, which is why it only fixes the colorings that are all the same color.

In the rotation of 180°, meanwhile, all elements in (13) are the same, while all elements in (24) are the same, which is why it fixes all colorings whose opposite vertices are the same color!

6. Pólya's Enumeration Theorem

Finally, we can propose Pólya's Enumeration Theorem in full:

Theorem 6.1 (Pólya's Enumeration Theorem (Multivariate))**.** *Let C^D be a set of colorings, and let G be a group action on C^D. The configuration generating function of C^D is equivalent to*

$$Z_G\left(\sum_{c \in C} w(c), \sum_{c \in C} w(c)^2, \sum_{c \in C} w(c)^3, \ldots\right)$$

Before we begin the proof, we provide a quick example.

Example. Let's conclude our example with the colored square up to rotation, with each vertex colored red r or blue b.

From before, the cycle index polynomial of C_4 acting on the vertices is

$$\frac{1}{4}(t_1^4 + t_2^2 + 2t_4)$$

Then, we just substitute $t_1 = r + b$, $t_2 = r^2 + b^2$, $t_4 = r^4 + b^4$, and we expand and simplify, yielding

$$b^4 + b^3 r + 2b^2 r^2 + br^3 + r^4,$$

as desired.

Now, we provide the proof in full:

Proof. If the set of configurations is F, recall that the CGF of $G \curvearrowright C^D$ is

$$\sum_{f \in F} W(f)$$

Instead of summing over configurations, let's sum over all the possible weights ω and counting how many configurations have weight ω

$$\sum_{f \in F} W(f) = \sum_{\omega} \omega \cdot |\{f \in F : W(f) = \omega\}|$$

$|\{f \in F : W(f) = \omega\}|$ is the number of configurations that have weight ω.

However, all elements in the same configuration have the same weight. If S_ω is the set of all *colorings* that have weight ω, then each configuration is contained in a single S_ω. In fact, each configuration is an orbit of S_ω!

Therefore, counting the number of total configurations with weight ω really just is counting the number of orbits in S_ω, stated as

$$|\{f \in F : W(f) = \omega\}| = |S_\omega / G|$$

This looks like something that Burnside's Lemma can count!

$$|S_\omega / G| = \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}_\omega(g)|$$

Here, $|\operatorname{Fix}_\omega(g)|$ refers to the number of fixed points of g that have a weight of ω. Now, let's bring it back to the CGF:

$$\text{CGF} = \sum_{f \in F} W(f) = \sum_{\omega} \omega \cdot |\{f \in F : W(f) = \omega\}| = \sum_{\omega} \omega \frac{1}{|G|} \sum_{g \in G} |\operatorname{Fix}_\omega(g)|$$

In classic Euler Circle fashion, we switch the double sums.

$$\text{CGF} = \frac{1}{|G|} \sum_{g \in G} \sum_{\omega} \omega |\operatorname{Fix}_\omega(g)|$$

Note that $\sum_\omega \omega |\operatorname{Fix}_\omega(g)|$ really is just counting each fixed point, multiplied by its weight, so

$$\sum_{\omega} \omega |\operatorname{Fix}_\omega(g)| = \sum_{x \in \operatorname{Fix}(g)} W(x)$$

Now, we can actually count how many fixed points there are for some permutation g using cycles. Recall that if g fixes a coloring x iff the colors in each cycle are the same (refer to 5 for an example).

Using generating functions, a cycle of length n has to have n of the same color, which is $\sum_{c\in C} w(c)^n$. Then, for each cycle in g, we can multiply the generating function for all of its cycles to get the weights of all its fixed points! This means

$$\sum_{x\in \text{Fix}(g)} W(x) = \prod_{\text{all cycles } y} \sum_{c\in C} w(c)^{|y|} = \left(\sum_{c\in C} w(c)^1\right)^{c_1(g)} \left(\sum_{c\in C} w(c)^2\right)^{c_2(g)} \cdots$$

Here, recall that $c_i(g)$ is the number of cycles of length i in g. Let's plug it back into our CGF:

$$\text{CGF} = \frac{1}{|G|} \sum_{g\in G} \left(\sum_{c\in C} w(c)^1\right)^{c_1(g)} \left(\sum_{c\in C} w(c)^2\right)^{c_2(g)} \cdots$$

But this is just our cycle index polynomial from before!

$$\text{CGF} = Z_G\left(\sum_{c\in C} w(c), \sum_{c\in C} w(c)^2, \sum_{c\in C} w(c)^3, \dots\right)$$

■

Example. Here's a more advanced example: find the number of graphs with 4 vertices up to isomorphism.

We turn this into a problem we can use PET on by coloring the edges of the graph either black or white: black means there is an edge there, and white means there is not.

Then, we have that D is the set of the $\binom{4}{2} = 6$ edges between the 4 vertices, and C is the set $\{\text{black}, \text{white}\}$. The group G will be the group of all permutations **on the vertices of the graph**, not on the edges. Thus, two graphs will be considered the same if you can rearrange their vertices and get the same graph.

Finding the Cycle Index Polynomial is often the hardest part. There are 24 elements in G.

The identity element keeps all 6 edges fixed, so it is t_1^6.

There are 6 elements in G that swap only 2 vertices. WLOG, say 1 and 2 are swapped. Then, the edges 13 and 23 are swapped, along with 14 and 24. Thus, there are 2 cycles of length 2 and 2 cycles of length 1, so this contributes $6t_1^2t_2^2$.

There are 8 elements in G that swap 3 vertices. WLOG, say the cycle is (123), with 1 going to 2, 2 to 3, and 3 to 1. The edges $14, 24, 34$ cycle, and the edges $12, 23, 31$ also cycle. There are 2 cycles of length 3, so this contributes $8t_3^2$.

There are then 6 elements that swap all 4 vertices in a cycle. WLOG, say the cycle is (1234). Then, $12, 23, 34, 14$ form a cycle, and $13, 24$ swap, so there is 1 cycle of length 4 and 1 cycle of length 2, contributing $6t_2t_4$.

Finally, there are 3 elements that swap 2 vertices with another 2 vertices, respectively. WLOG, say $1, 2$ are swapped and $3, 4$ are swapped. Then, the edges 13 and 24 swap, and the edges 14 and 23 do too. Thus, this contributes $3t_1^2t_2^2$.

Adding them all up and dividing by $|G| = 24$, we get the cycle index polynomial, which is

$$\frac{1}{24}\left(t_1^6 + 9t_1^2t_2^2 + 8t_3^2 + 6t_2t_4\right)$$

Finally, we can use PET to get the configuration generating function. Plugging in $t_i = b^i + w^i$ and working through the algebra with a computer, we get

$$b^6 + b^5w + 2b^4w^2 + 3b^3w^3 + 2b^2w^4 + bw^5 + w^6$$

There is 1 graph each with 6, 5, 1, or 0 edges. There are 2 graphs each with 2 and 4 edges, and there are 3 graphs with 3 edges.

Remark 6.2. By setting all the weights of the colors to 1, the number that the CGF becomes is simply the total number of configurations.

7. Univariate Pólya's Enumeration Theorem

There also is a variation of Pólya's Enumeration Theorem where we actually do assign numeric weights to colors. Let the weight of a color c be $w(c)$, a non-negative integer. If the set of all colors is C, let its generating function be $C(t) = \sum_{c \in C} t^{w(c)}$.

Definition 7.1 (Pólya's Enumeration Theorem (Univariate))**.** Let C^D be a set of colorings, and let G be a group action on C^D. If F is the set of all configurations,

$$\sum_{f \in F} t^{W(f)} = Z_G\left(C(t), C(t^2), C(t^3), \ldots\right)$$

This theorem can easily be derived from the multivariate case by setting $w(c) = t^i$, where i is the numeric weight of c.

Example. Our example from before with the number of graphs up to isomorphism could have worked just as well by using univariate PET. Instead of assigning the colors' weights as b and w, we could've assigned them weights of 1 and 0 (since black is 1 edge and white is 0), giving t and 1 as weights respectively. The final CGF we end up with becomes

$$t^6 + t^5 + 2t^4 + 3t^3 + 2t^2 + t + 1$$

an arguably simpler expression.

Example. Now, let's find the multigraphs with 4 vertices and up to 2 edges between nodes, up to isomorphism.

Note that we still have D as the set of edges and G as the permutation group of the vertices; it is only the colors that will change. Say we have the colors white, gray, and black. Black will represent 2 edges, and will have a weight of 2. Gray will represent 1 edge, and will have a weight of 1. Then, of course, white is no edge, and has a weight of 0.

First, we find $C(t)$. Easily, this is just

$$C(t) = 1 + t + t^2$$

Then, we must find the cycle index polynomial. But we've already done this! Since we only changed the colors, we can use the same CIP from before, which is

$$\frac{1}{24}\left(t_1^6 + 9t_1^2t_2^2 + 8t_3^2 + 6t_2t_4\right)$$

Finally, plugging in $t_i = C(t^i)$, we get

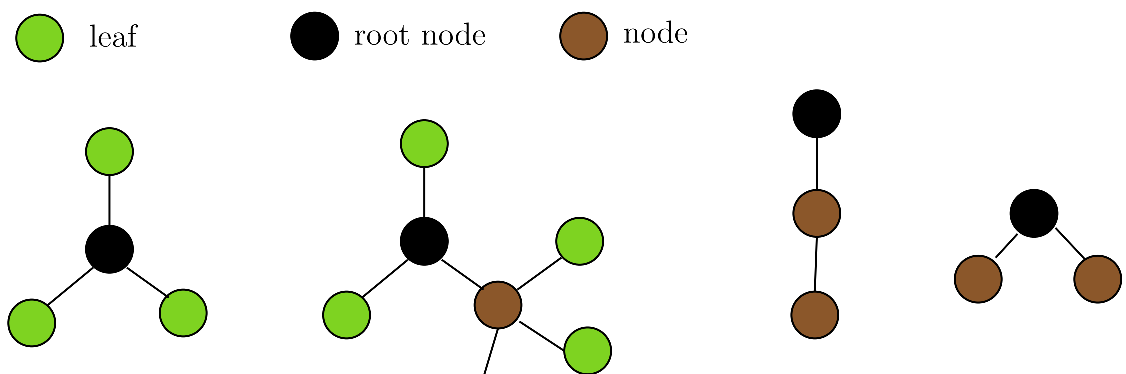
$$t^{12} + t^{11} + 3t^{10} + 5t^9 + 8t^8 + 9t^7 + 12t^6 + 9t^5 + 8t^4 + 5t^3 + 3t^2 + t + 1$$

For example, there will be 5 multigraphs with 9 edges. However, note that it does not tell us how many double edges and how many single edges there are; if we wanted to know that, we'd have to use the multivariate case.

8. Applications

8.1. **Rooted Trees.** The univariate case of PET is very powerful because the generating function $C(t)$ is flexible. $C(t)$ is the generating function of the ways to color a single node, and usually, the node can only be colored one color, thus $C(t) = \sum_{c \in C} t^{w(c)}$.

Now, consider the class of non-plane ternary rooted trees $\mathcal{F}$, where every node can have up to 3 children. For the purposes of Pólya's enumeration theorem, we actually force all nodes to either be leaves (0 children) or have 3 children. However, leaves will have a size of 0, and by ignoring the leaves, we get non-plane ternary rooted trees, as desired.



1 node

2 node

3 nodes
(leaves excluded)

For every node, we have that it can either be a leaf with size 0, or be a node with size 1 along with 3 children. However, these 3 children can be permuted, and we'd still have the same ternary tree. This is a job for Pólya's Enumeration Theorem!

Consider 3 child trees up to permutation. There are 6 permutations, and the cycle index polynomial of these 6 permutations turns out to be

$$\frac{1}{6}\left(t_1^3 + 3t_1t_2 + 2t_3\right)$$

Thus, we have that if a node has 3 children, the generating function for its children is

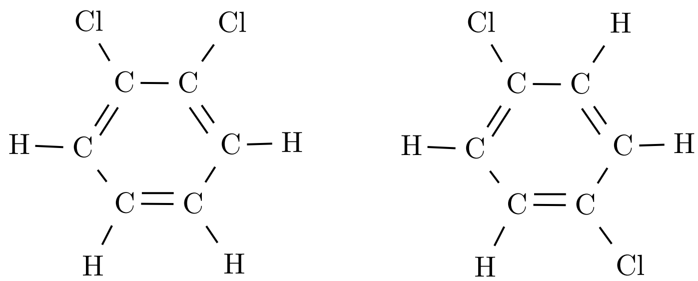
$$\frac{1}{6}\left(F(t)^3 + 3F(t)F(t^2) + 2F(t^3)\right).$$

Now, we know that a node can either be a leaf of size 0, or 1, or be a node of size 1, t, along with these 3 children. Thus, we can recursively define the class of ternary rooted trees $F(t)$:

$$F(t) = 1 + t \cdot \frac{1}{6}\left(F(t)^3 + 3F(t)F(t^2) + 2F(t^3)\right)$$

From this, a recurrence for the coefficients of $F(t)$ can be derived. The first few coefficients are $1, 1, 1, 2, 4, 8, 17, 39, 89, 211, 507, 1238, 3057, 7639, 19241$.

8.2. **Pólya's Isomer Method.** Pólya's theorem can famously be used to count the number of isomers of a chemical formula. A chemical formula, like $C_6H_4Cl_2$, specifies how much of each element are in the compound. However, since compounds exist in 3-D space, more than one compound can correspond to the same formula if they have different structures. These different compounds are called isomers.



Two Distinct Isomers of $C_6H_4Cl_2$

Because of how chemical bonds work, in the chemical formulas $C_6H_xCl_{6-x}$ for $x \in \{0, 1, 2, 3, 4, 5, 6\}$, the carbons form a regular hexagon and the hydrogens and chlorines each bond to a carbon. This is reminscent of counting necklaces: it's really a question of how to color the 6 corners of a hexagon 2 different colors (one for hydrogen and one for chlorine) up to rotation and reflection.

First, we find the cycle index polynomial. There are 6 rotations and 6 reflections of this hexagon, which is the dihedral group of order 12. Upon inspection, we get that the cycle index polynomial is

$$Z_G = \frac{1}{12}\left(t_1^6 + 4t_2^3 + 2t_3^2 + 2t_6 + 3t_1^2 + t_2^2\right)$$

Then, if our colors are H and Cl for hydrogen and chlorine respectively, we get that the generating function is

$$H^6 + H^5Cl + 3H^4Cl^2 + 3H^3Cl^3 + 3H^2Cl^4 + HCl^5 + Cl^6$$

Now, we know that there are 3 isomers of $C_6H_4Cl_2$, since the coefficient of H^4Cl^2 is 3. In fact, we know the number of isomers of all such $C_6H_xCl_{6-x}$. Although this could have easily been found by careful inspection, Pólya's Enumeration Theorem can be used to enumerate much more complex isomers.

8.3. **Cyclic Generating Functions.** PET gives a concise solution to problem 4 of week 1's problem set, which asks if $\mathcal{B} \cong \textsc{Cyc}\,\mathcal{A}$, where both classes are unlabelled, then what is $A(z)$?

We first note that we should consider the group of rotations of a regular n-gon, which is called the cyclic group of order n and denoted C_n. Then, for every size n, $\textsc{Cyc}_n\,\mathcal{A}$ will just be the number of ways to have n elements from $\mathcal{A}$ up to rotation by the C_n group.

Now, we can find its cycle index polynomial. Say the rotation of $\frac{2\pi}{n}$, or the rotation of 1 vertex, is r. Then, note that the rotation of kr will have $\gcd(k, n)$ cycles of size $\frac{n}{\gcd(k,n)}$ in its permutation. (To see this, label the vertices of the n-gon 1 to n. The vertices that are the same mod $\gcd(k,n)$ will form a cycle.) This gives the cycle index polynomial

$$\frac{1}{n}\sum_{k=1}^{n} t_{n/\gcd(k,n)}^{\gcd(k,n)}$$

Fortunately, this can be simplified. By summing instead of the possible gcd's, we have that the cycle index polynomial is

$$\frac{1}{n}\sum_{d|n}\phi(d)t_d^{n/d}$$

Now, note that if $\mathcal{B}_n \cong \textsc{Cyc}_n(\mathcal{A})$, then we would have

$$B_n(z) = \frac{1}{n}\sum_{d|n}\phi(d)(A(z^d))^{n/d}$$

To get $B(z)$, we simply sum over all $n \geq 1$:

$$B(z) = \sum_{n=1}^{\infty}\frac{1}{n}\sum_{d|n}\phi(d)(A(z^d))^{n/d}$$

As always in Euler Circle, we switch the summations.

$$B(z) = \sum_{d=1}^{\infty}\sum_{d|n}\frac{1}{n}\phi(d)(A(z^d))^{n/d}$$

We set $n = kd$ to finally get

$$B(z) = \sum_{d=1}^{\infty}\frac{\phi(d)}{d}\sum_{k=1}^{\infty}\frac{1}{k}(A(z^d))^k = \sum_{d=1}^{\infty}\frac{\phi(d)}{d}\log\frac{1}{1-A(z^d)}$$

8.4. **Number Theory.** We end our discussion of PET's applications with a surprising application in number theory. We consider the cyclic group C_n on the regular n-gon. Formally, we have that the set $X = \{1,\dots,n\}$ is the set of its vertices. Then, let $C = \{1,\dots,a\}$ be the colors.

We recall that the cycle index polynomial of C_n on the vertices of the n-gon is simply

$$\frac{1}{n}\sum_{d|n}\phi(d)t_d^{n/d}$$

Now, what if we set the weights of all the colors to just be 1? Recall that this just means we count the total number of configurations of coloring the vertices of a n-gon in one of a colors, up to rotation. When the weights of all the colors are 1, then t_i is just a, so we have

$$\frac{1}{n}\sum_{d|n}\phi(d)a^{n/d}$$

However, note that this must be an integer, since it's counting configurations! This means that, for any integers a and n

$$\sum_{d|n}\phi(d)a^{n/d} \equiv 0 \pmod{n}$$

This is actually a generalization of Fermat's little theorem, along with Gauss's theorem that states $\sum_{d|n} \phi(d) = n$. It's quite extraordinary that PET gives such a result in number theory.

9. Conclusion

Pólya's Enumeration Theorem is a very powerful technique to count the number of colorings up to some group action. Pólya's original paper that published his enumeration theorem was unique in the sense that it only contained the one theorem. However, it ran over 100 pages detailing numerous applications, far more than we could ever even skim through here.

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