FROBENIUS PROBLEM AND THE RESTRICTED PARTITION FUNCTION

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Abstract. This paper is an exposition of the Frobenius number problem and the restricted partition function. In this note we compute the Frobenius number $g(a, b)$ and the restricted parttiton function in terms of Fourier-Dedekind sums. This paper is a part of Generating function class of 2021 and serves as a resource where students can learn from.

1. Introduction

The restricted partition function is defined by

$$p_{\mathbf{a}}(n) = \#\left\{(x_1, x_2, \ldots, x_r) \in \mathbb{N}^r \middle| \sum_{i=1}^{r} a_i x_i = n\right\}$$

for fixed integer vector $(x_1, x_2, \ldots, x_r) \in \mathbb{N}^r$. It is known that ([1], Theorem 1.8)

$$p_{\mathbf{a}}(n) = \frac{(-1)^r}{(r-1)!} B_r^X(-n) + s_{-n}(a_2, a_3, \ldots, a_r; a_1) + s_{-n}(a_3, a_4, \ldots, a_1; a_2) + \ldots +$$

$$s_{-n}(a_1, a_2, \ldots, a_{r-1}; a_r)$$

where

$$s_n(a_1, \ldots, a_r; b) := \frac{1}{b} \sum_{j=1}^{b-1} \frac{\epsilon_b^{jn}}{(1 - \epsilon_b^{ja_1}) \ldots (1 - \epsilon_{x_b})^{ja_r}}$$

whenever $a_1, \ldots, a_r$ are pairwise relatively prime. The Frobenius problem asks us to find the largest positive integer n such that $p_{\mathbf{a}}(n) = 0$ and is called the *Frobenius number* and denote it by $g(a_1, \ldots, a_r)$. Let $V_{(a_1, a_2, \ldots, a_r)} = \{i | i = m_1 a_1 + m_2 a_2 + \ldots + m_r a_r \, , m_j > 0\}$. V is a semigroup as it neither has inverses nor an identity element but has closure and associativity with respect to addition. Also note that $p_{\mathbf{a}}(\min(V_{(a_1, a_2, \ldots, a_r)}) - 1) = 0$ and $p_{\mathbf{a}}(\min(V_{(a_1, a_2, \ldots, a_r}) + k) > 0$ for all $k \in \mathbb{N}_0$.

There is a geometric interpretation of restricted partition function. Let

$$\mathcal{G} = \{(a_1, \ldots, a_d) \in \mathbb{R}^d \,|\, a_i \geq 0 \,\forall\, 1 \geq i \geq d \,, x_1 a_1 + \ldots + x_d a_d = 1\}$$

The n^{th} dilate of a set $S \subset \mathbb{R}^d$ is defined as

$$\{(na_1, \ldots, na_d) \,|\, (a_1, \ldots, a_d) \in \mathbb{R}^d\}$$

The restricted partition function counts precisely the non-negative integer lattice points in the n^{th} dilate $\mathcal{G}$. The set $\mathcal{G}$ turns out to be a polytope.

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2. Existence of Frobenius number and Partial fraction decomposition

In this section we prove that Frobenius number exists and uniqueness of partial fraction expansion of a rational function.

Theorem 2.1. *Given* $(a_1, a_2, \dots, a_k)$ *such that* $\gcd(a_1, a_2, \dots, a_k) = 1$ *then exists an integer* N *such that any integer* $s \geq N$ *is representable as a non-negative integer combination of* $a_1, \dots, a_n$.

Proof. Since $\gcd(a_1, a_2, \dots, a_k) = 1$ we can write $m_1a_1 + m_2a_2 + \dots + m_ka_k = 1$ for some integers m_i. Denote by P and $-Q$ the sum of positive terms and negative terms in this decomposition respectively, P and Q belong to the semigroup $V_{(a_1,a_2,\dots,a_k)}$. We now have $P - Q = 1$. Any integer $r \geq 0$ can be written as $sa_1 + t$ where $0 \leq t < a_1$. We now have $a_1(Q-1) + r = sa_1 + (a_1 - 1 - t)Q + tP$ which is also in the semigroup $V_{(a_1,a_2,\dots,a_k)}$. Thus $q \in V_{(a_1,a_2,\dots,a_k)}$ for all $q \geq x_1(Q-1)$. ∎

The above proof implies $\min\{n|p_{\mathbf{a}}(n) = 0\} < \min a_1, a_2, \dots, a_k(Q-1)$. However this bound can be largely improved.

We now prove uniqueness of partial fraction decomposition of rational functions which we use to give an alternate way to compute $p_{a_1,a_2}(n)$. To prove uniqueness of partial fraction decomposition of rational functions $F = \frac{n(z)}{r(z)}$ it suffices to consider the case $\deg(n(z)) \leq \deg r(z)$ as we can always reduce the degree of $n(z)$ by division algorithm.

Theorem 2.2. *Given a rational function* $P(z) = \frac{f(z)}{\prod_{i=1}^{n}(z-a_i)^{e_1}}$ *where* $\sum_{i=1}^{n} e_i \geq \deg(f)$ *and* $a_i \in \mathbb{C}$ *then there exists a decomposition*

$$P(z) = \sum_{i=1}^{n}\left(\sum_{k=1}^{e_i} \frac{b_{k,i}}{(z-a_i)^k}\right)$$

where $b_{k,i} \in \mathbb{C}$ *are unique.*

To prove the above theorem we first need some intermediate lemmas.

Lemma 2.3. *If* $f, g \in \mathbb{C}[x]$ *where* $g \neq 0$. *Suppose that* $g = ab$ *where* $a, b \in \mathbb{C}[x]$ *and* $\gcd(a, b) = 1$ *then* $\frac{f}{g} = \frac{c}{a} + \frac{d}{b}$ *for some* $c, d \in \mathbb{C}[x]$

Proof. Since $\mathbb{C}[x]$ is a euclidean domain, there exist $k, l \in \mathbb{C}[x]$ such that $ak + bl = 1$. Thus $f = afk + bfl$ and so

$$\frac{f}{g} = \frac{afk + bfl}{ab} = \frac{fk}{b} + \frac{fl}{a}$$

So $fl = c$ and $fk = b$ are the required elements of $\mathbb{C}[x]$ ∎

Lemma 2.4. *If* $f, g \in \mathbb{C}[x]$ *where* $g \neq 0$ *and* $g = a_1^{k_1}a_2^{k_2}\dots a_l^{k_l}$ *where* $a_i \in \mathbb{C}[x]$, a_i *are pairwise relatively prime and* $k_j \in \mathbb{N}$. *Then*

$$\frac{f}{g} = \frac{c_1}{a_1^{k_1}} + \frac{c_2}{a_2^{k_2}} + \dots + \frac{c_l}{a_l^{k_l}}$$

for some $c_1, c_2, \dots, c_l \in \mathbb{C}[x]$.

Proof. Since a_1 is relatively prime with each of a_i where $2 \geq i \geq n$, applying lemma 2.4 we have

$$\frac{f}{g} = \frac{c_1}{a_1^{k_1}} + \frac{b_1}{a_2^{k_2} \dots a_l^{k_l}}$$

It suffices to iterate this process again. ∎

Proof of Theorem 2.2

Proof. With lemma 2.4 it suffices to prove that

$$\frac{F}{g^k} = q + \frac{r_1}{g} + \frac{r_2}{g^2} + \dots + \frac{r_k}{g^k}$$

where $F, g, q, r_1, r_2, \dots, r_k \in \mathbb{C}[x]$. Since $\mathbb{C}[x]$ is Euclidean , we can write $F = gq_k + r_k$ where $\deg(r_k) < \deg(g)$. Now applying division algorithm again we have $q_k = gq_{k-1} + r_{k-1}$ where $\deg(r_{k-1} \leq \deg(q_{k-1}))$. Continuing in this fashion,

$$\frac{F}{g^k} = \frac{gq_k + r_k}{g^k} = \frac{r_k}{g^k} + \frac{g^2 q_{k-1} + g r_{k-1}}{g^k} = \dots = q + \frac{r_1}{g} + \frac{r_2}{g^2} + \dots + \frac{r_k}{g^k}$$

∎

Note that division by $x - a_i \in \mathbb{C}[x]$ to any polynomial has a remainder which is a complex number.

3. Computing $g(a, b)$

The main result we prove in this section is

Theorem 3.1. *If a_1 and a_2 are relatively prime positive integers, then*

$$g(a, b) = ab - a - b$$

We begin be seeing that

$$\frac{1}{(1 - z^a)(1 - z^b)} = \sum_{t=0}^{\infty} p_{a,b}(t) z^t$$

Thus,

$$p_{a,b}(t) = [z^0] \frac{1}{(1 - z^a)(1 - z^b) z^t}$$

Computing the restricted partition function by partial fractions is illustrated by following example.
Note that

$$\left[z^{-1}\right] \frac{1}{z^{t+1}(1 - z^a)(1 - z^b)} = p_{a,b}(t)$$

Also

$$\oint_{|z|=R} \frac{1}{z^{t+1}(1 - z^a)(1 - z^b)}$$

tends to 0 as $R \to \infty$ as

$$\left| \oint_{|z|=R} \frac{1}{z^{t+1}(1 - z^a)(1 - z^b)} \right| \leq \frac{2\pi R}{R^{t+1}}$$

Denote $\epsilon_a = e^{\frac{2i\pi}{a}}$. By residue theorem,

$$\mathrm{Res}_{z=0}\frac{1}{(1-z^a)(1-z^b)z^{t+1}}+\sum_{k=0}^{a-1}\mathrm{Res}_{z=\epsilon_a^k}\left(\frac{1}{(1-z^a)(1-z^b)z^{t+1}}\right)+\sum_{k=0}^{b-1}\mathrm{Res}_{z=\epsilon_b^k}\left(\frac{1}{(1-z^a)(1-z^b)z^{t+1}}\right)=0$$

However $N(t) = \mathrm{Res}_{z=0}\frac{1}{(1-z^a)(1-z^b)z^{t+!}}$. As we have simple poles at a$^{\text{th}}$ and b$^{\text{th}}$ roots of unity except we have a pole of order 2 at $z = 1$. We can compute

$$\mathrm{Res}_{z=\epsilon_a^k}\left(\frac{1}{(1-z^a)(1-z^b)z^{t+1}}\right) = \frac{1}{a}\lim_{z\to\epsilon_a^k}\frac{1}{(1-z^b)z^{t+1}} = \frac{1}{a}\frac{\epsilon_a^{-(t+1)b}}{1-\epsilon_a^{kb}}$$

$$\mathrm{Res}_{z=1}\left(\frac{1}{(1-z^a)(1-z^b)z^{t+1}}\right) = \lim_{z\to1}\frac{d}{d\mathrm{z}}(1-z)^2\frac{1}{(1-z^a)(1-z^b)z^{t+1}} = \frac{t}{ab}+\frac{1}{2a}+\frac{1}{2b}$$

Thus

$$N_{\mathbf{m}}(t) = \frac{t}{ab}+\frac{1}{2a}+\frac{1}{2b}+\frac{1}{a}\sum_{k=1}^{a-1}\frac{\epsilon_a^{-(t)b}}{\epsilon_a^{kb}-1}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{\epsilon_b^{-(t)a}}{\epsilon_b^{ka}-1}$$

Another way to see this would be using partial fraction expansion. Let $p(z) = \frac{1}{(1-z^a)(1-z^b)z^t}$ and suppose

$$p(z) = \sum_{i=1}^{t}\frac{A_i}{z^i}+\frac{B_1}{z-1}+\frac{B_2}{(z-1)^2}+\sum_{k=1}^{a-1}\frac{C_k}{z-\epsilon_a^k}+\sum_{j=1}^{b-1}\frac{C_j}{z-\epsilon_b^j}$$

As we are interested in the constant term we can ignore the principal part of the partial fraction expansion. To obtain B_2 we multiply $p(z)$ by $(z-1)^2$ and take the limit as $z\to1$ to get

$$\lim_{z\to1}(1-z)^2\frac{1}{(1-z^a)(1-z^b)z^t} = \frac{1}{ab}$$

For computing B_1 we subtract the term of pole of order two from $p(z)$ and take the limit as $z\to1$ to get

$$\lim_{z\to1}(1-z)\left(\frac{1}{(1-z^a)(1-z^b)z^t}-\frac{1}{ab(1-z)^2}\right) = \frac{1-t}{ab}-\frac{1}{2a}-\frac{1}{2b}$$

To calculate C_k multiply $p(z)$ by $\epsilon_a^k - z$ and take the limit as $z\to\epsilon_a^k$ to get

$$\lim_{z\to\epsilon_a^k}(\epsilon_a^k-z)\frac{1}{z^t(1-z^a)(1-z^b)} = \lim_{z\to\epsilon_a^k}\frac{\epsilon_a^k}{az^{a-1}}\lim_{z\to\epsilon_a^k}\frac{1}{z^t(1-z^b)} = \frac{1}{a(\epsilon_a^{kb}-1)\epsilon_a^{k(t-1)}}$$

Similarly,

$$D_k = \frac{1}{a(\epsilon_b^{ka}-1)\epsilon_b^{k(t-1)}}$$

However $R(a,b)$ being constant term of $p(z)$ we get

$$R(a,b) = \left(\frac{B_1}{z-1}+\frac{B_2}{(z-1)^2}+\sum_{k=1}^{a-1}\frac{C_k}{z-\epsilon_a^k}+\sum_{j=1}^{b-1}\frac{C_j}{z-\epsilon_b^j}\right)\Bigg|_{x=0}$$

$$= \frac{t}{ab}+\frac{1}{2a}+\frac{1}{2b}+\frac{1}{a}\sum_{k=1}^{a-1}\frac{\epsilon_a^{-kt}}{\epsilon_a^{kb}-1}+\frac{1}{b}\sum_{k=1}^{b-1}\frac{\epsilon_b^{-kt}}{\epsilon_b^{ka}-1}$$

If $b = 1$ one of the sum vanishes and we have

$$N_{a,1}(t) = \frac{t}{a} + \frac{1}{2a} + \frac{1}{2} + \frac{1}{a}\sum_{k=1}^{a-1} \frac{\epsilon_a^{-kt}}{\epsilon_a^{kb} - 1}$$

However

$$\begin{aligned} N_{a,1}(t) &= |\{(m_1, m_2)|(m_1, m_2) \in \mathbb{Z}^2_{\geq 0}, x - 1m_2 + m_2 = t\} \\ &= |\{m_1 \in \mathbb{Z}|0 \geq m_1 \geq \frac{t}{a}\}| \\ &= \left\lfloor \frac{t}{a} \right\rfloor + 1 \end{aligned}$$

Thus we have

$$\frac{1}{a}\sum_{k=1}^{a-1} \frac{\epsilon_a^{-kt}}{\epsilon_a^{kb} - 1} = \frac{1}{2} - \frac{1}{2a} - \left\{\frac{t}{a}\right\}$$

where $\left\{\frac{t}{a}\right\} = \frac{t}{a} - \left\lfloor \frac{t}{a} \right\rfloor$ As $\gcd(a, b) = 1$ there exists $b^{-1} \in \mathbb{Z}/a\mathbb{Z}$ such that $bb^{-1} \cong 1 \pmod a$. Also b being a unit the sets $\mathbb{Z}/a\mathbb{Z}$ and $b^{-1}\mathbb{Z}/a\mathbb{Z}$ are same. Thus

$$\frac{1}{a}\sum_{k=1}^{a-1} \frac{\epsilon_a^{-kt}}{\epsilon_a^{kb} - 1} = \frac{1}{a}\sum_{k=1}^{a-1} \frac{\epsilon_a^{-ktb^{-1}}}{\epsilon_a^{k} - 1}$$

which says

$$\frac{1}{a}\sum_{k=1}^{a-1} \frac{\epsilon_a^{-kt}}{\epsilon_a^{kb} - 1} = \frac{1}{2} - \left\{\frac{ab^{-1}}{t}\right\} - \frac{1}{2a}$$

After substitution this yields a formula due to Peter Barlow and Tiberiu Popoviciu .

Theorem 3.2. *If a, b are relatively prime than*

$$N_{a,b}(t) = \frac{t}{ab} - \left\{\frac{tb^{-1}}{a}\right\} - \left\{\frac{ta^{-1}}{b}\right\} + 1$$

where $bb^{-1} \cong 1 \pmod a$ and $aa^{-1} \cong 1 \pmod b$

Now we prove an intermediate result which we will use to compute $R(a, b)$.

Proposition 3.3. *If a, b are relatively prime positive integer and $t \in [ab - 1]$ and is not a multiple of a or b then*

$$N_{a,b}(t) + N_{a,b}(ab - n) = 1$$

Proof. Using Theorem 3.1,

$$\begin{aligned} N_{a,b}(ab - t) &= \frac{ab - t}{ab} - \left\{\frac{(ab - t)b^{-1}}{a}\right\} - \left\{\frac{(ab - t)a^{-1}}{b}\right\} + 2 \\ &= 2 - \frac{t}{ab} - \left\{-\frac{ta^{-1}}{b}\right\} - \left\{-\frac{tb^{-1}}{a}\right\} \\ &= \frac{-t}{ab} + \left\{\frac{tb^{-1}}{a}\right\} + \left\{\frac{ta^{-1}}{b}\right\} \\ &= 1 - N_{a,b}(t) \end{aligned}$$

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Proof of Theorem 3.1 From Proposition 3.3 and $p_{a,b}(a+b)=1$ we have proved $g(a,b)=ab-a-b$. Noting that $\left\{\frac{m}{a}\right\} \leq 1-\frac{1}{a}$ and from Theorem 3.2 we have

$$p_{a,b}(ab-a-b+n) \geq \frac{ab-a-b}{n} - \left(1-\frac{1}{a}\right) - \left(1-\frac{1}{b}\right) + 1 = \frac{n}{ab} > 0$$

Note that we have proved even more. By Proposition 3.3 , exactly half of integers in $[ab-1]$ that are not divisible by a or b are representable and representable integers less than ab have a unique representation. This is sylvester's theorem.

Theorem 3.4 (Sylvester's Theorem)**.** *Let a, b be relatively prime integers. Exactly half of integers between 1 and $(a-1)(b-1)$ are representable by a, b.*

4. Computing Restricted partition function

For reasons of simplicity we assume here $\gcd(a_i, a_j) = 1$. Note that with this assumption all poles except $z = 1$ are simple of the generating function of $p_{\mathbf{a}}(t)$. In similar fashion to previous section,

$$\sum_{t=0}^{\infty} N_{\mathbf{m}}(t) z^t = \frac{1}{(1-z^{a_1})\dots(1-z^{a_d})}$$

and so

$$\mathrm{Res}_{z=0}\left(\frac{1}{z^{t+1}(1-z^{a_1})\dots(1-z^{a_d})}\right) = N_m(t)$$

Also

$$\oint_{|z|=R} \frac{1}{z^{t+1}(1-z^{a_1})\dots(1-z^{a_d})} dz$$

tends to 0 as $R \to \infty$ by the same argument as given in previous section. Let

$$f(z) = \frac{1}{z^{t+1}(1-z^{a_1})\dots(1-z^{a_d})}$$

then we have

$$\sum_{a_i \in X} \sum_{j=1}^{a_i-1} \mathrm{Res}_{z=\epsilon_{a_i}^j}(f(z)) + \mathrm{Res}_{z=1}(f(z)) + \mathrm{Res}_{z=0}(f(z)) = 0$$

where $X = \{a_1, \dots, a_d\}$.

We now compute

$$\mathrm{Res}_{z=\epsilon_{a_i}^k} f(z) = \lim_{z \to \epsilon_{a_i}^k} (\epsilon_{a_i}^k - z) f(z) = \frac{\epsilon_{a_i}^{-(t+1)k}}{a_i \prod_{y \in X - \{a_i\}} (1-\epsilon_{a_i}^{ky})}$$

To compute the residue at 1 we have

$$\mathrm{Res}_{z=1} f(z) = \mathrm{Res}_{z=0} e^z f(e^z)$$

However note that

$$\frac{z^d e^{xz}}{(e^{a_1 z}-1)\dots(e^{a_d z}-1)} = \sum_{k=0}^{\infty} B_k^X(x) \frac{z^k}{k!}$$

Where B_k^X are called as *Bernoulli-Barnes polynomial.* Thus

$$\mathrm{Res}_{z=0} e^z = \frac{(-1)^d}{(d-1)!} B_d^X(-t)$$

Let us define

$$s_n(a_1, a_2, \ldots, a_{d-1}; a_d) = \frac{1}{t}\sum_{k=1}^{n-1} \frac{\epsilon_{a_d}^{kn}}{(1-\epsilon_{a_d}^{kx_1})\ldots(1-\epsilon_{a_{d-1}})^{kn}}$$

These are called as *Fourier-Dedekind sums.* In this notation,

$$p_{\mathbf{a}}(t) = \frac{(-1)^d}{(d-1)!}B_d^X(-t)+s_{-n}(a_2, a_3, \ldots, a_d; a_1)+s_{-n}(a_3, a_4, \ldots, a_1; a_2)+\ldots+s_{-n}(a_1, a_2, \ldots, a_{d-1}; a_d)$$

The term $\frac{(-1)^d}{(d-1)!}B_d^X(-t)$ is also called as polynomial part of the restricted partition function denoted $\text{poly}_A(n)$.

Example. Computing Bernoulli Banes polynomial for $d = 3, 4$ yields

$$N_{a,b,c}(t) = \frac{t^2}{2abc} + \frac{t}{2}\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right) + \frac{1}{12}\left(\frac{3}{a}+\frac{3}{b}+\frac{3}{c}+\frac{a}{bc}+\frac{b}{ac}+\frac{c}{ab}\right)$$
$$+s_{-t}(b, c; a) + s_{-t}(c, a; b) + s_{-t}(a, b; c)$$

and

$$N_{a,b,c,d}(t) = \frac{n^3}{6abcd} + \frac{n^2}{4}\left(\frac{1}{abc}+\frac{1}{abd}+\frac{1}{acd}+\frac{1}{bcd}\right)$$
$$+\frac{n}{12}\left(\frac{3}{ab}+\frac{3}{ac}+\frac{3}{ad}+\frac{3}{bc}+\frac{3}{bd}+\frac{3}{ad}+\frac{a}{bcd}+\frac{b}{cda}+\frac{c}{dab}+\frac{d}{abc}\right)$$
$$+\frac{1}{24}\left(\frac{a}{bc}+\frac{a}{bd}+\frac{a}{cd}+\frac{b}{ac}+\frac{b}{ad}+\frac{b}{cd}+\frac{c}{ab}+\frac{c}{ad}+\frac{c}{bd}+\frac{d}{ab}+\frac{d}{ac}+\frac{d}{bc}\right)$$
$$+s_{-t}(a, b, c; d) + s_{-t}(b, c, d; a) + s_{-t}(c, d, a; b) + s_{-t}(d, a, b; c)$$

Following examples illustrate how to compute the restricted parttiton function using partial fractions.

Example. We may compute $p_{1,2}(m)$.

$$\frac{1}{(1-z^2)(1-z^3)} = \frac{1}{4}\left(\frac{1}{1+z}+\frac{1}{1-z}+\frac{2}{(1-z)^2}\right)$$
$$= \frac{1}{4}\left(\sum_{m=0}^{\infty}(-1)^m z^m + \sum_{z=0}^{\infty} z^m + 2\sum_{z=0}^{\infty}(m+1)z^m\right)$$

which gives us $p_{1,2}(m) = \frac{1}{4}(2m+3+(-1)^m)$

Example. We now compute $p_{1,2,3}(n)$

$$\frac{1}{(1-z)(1-z^2)(1-z^3)} = \frac{1}{6(1-z)^3} + \frac{1}{4(1-z)^2} + \frac{1}{4(1-z^2)} + \frac{1}{3(1-z^3)}$$
$$= \frac{1}{6}\sum_{m=0}^{\infty}\frac{(m+1)(m+2)}{2}z^m + \frac{1}{4}\sum_{m=0}^{\infty}(m+1)z^m + \frac{1}{4}\sum_{m=0}^{\infty}z^{2m} + \frac{1}{3}\sum_{m=0}^{\infty}z^{3m}$$

and this is equivalent to

$$p_{1,2,3}(m) = \left\lfloor \frac{m^2+6m+5}{12} \right\rfloor$$

5. Bounds on restricted partition function

Fourier dedekind sums are difficult to compute. In next section we outline some results about reciprocity of Fourier dedekind sums. The knowledge of an exact formula for $p_{\mathbf{a}}(n)$ is very important in many branches of mathematics. It is not surprising that finding formulas for restricted parttition functionis very difficult since determining whether $p_{\mathbf{a}}(n) > 0$ is an $\mathcal{NP}$ complete problem. Thus approximation formulas for $p_{\mathbf{a}}(n)$ are of great interest. Below is an instance of such a bound.

Theorem 5.1. *Let $a_1, a_2, \dots, a_m$ be such that* $\gcd(a_1, \dots, a_m) = 1$. *Then*

$$p_{x_1,\dots,x_m}(t) \sim \frac{t^{m-1}}{a_1 a_2 \dots a_m (m-1)!}$$

as $x \to \infty$.

Proof. Consider the generating function

$$f(z) = \frac{1}{(1-z^{a_1})(1-z^{a_2})\dots(1-z^{a_m})} = \sum_{t=0}^{\infty} p_{a_1,\dots,a_m}(t) z^t$$

As stated before all poles of $f(z)$ lie on the unit circle and the pole $z = 1$ has multiplicity m where as other poles which are roots of unity (i.e. $\omega = e^{\frac{2i\pi k}{a_i}}$). Indeed order of pole ω will be number of a_j such that $a_i \mid ka_j$ which is strictly less than m as $\gcd(a_1, a_2, \dots, a_m) = 1$. So there will be a term in the $\frac{c}{(1-z)^m}$ partial fraction expansion of f. Coefficient of z^t in $\frac{c}{(1-z)^m}$ is $c\binom{t+m-1}{m-1}$. All other terms in the partial fraction expansion will be of the form $\frac{b}{(1-\omega z)^j}$ which contribute $b \times \omega^j \binom{m+j}{j-1}$. The total sum of all these terms is negligible to $c\binom{t+m-1}{m-1}$ since $j < m$ and as $t \to \infty$, we have $p_{a_1,\dots,a_m}(t) \sim c\binom{t+m-1}{m-1}$ or $p_{a_1,\dots,a_m}(t) \sim c\frac{m^{t-1}}{(m-1)!}$. From the partial fraction expansion,

$$f(z) = \frac{1}{(1-z)^m} + O((1-z)^{-m+1})$$

or

$$\frac{(1-z)\dots(1-z)}{(1-z^{x_1})\dots(1-z)^{x_m}} = c + (1-z)^m O(z^{-m+1})$$

and by L'hopital's rule $\lim_{z\to\infty} \frac{1-z}{1-z^{a_i}} = \frac{1}{a_i}$ whereas $(1-z)^m O(z^{-m+1}) \to 0$ as $z \to \infty$. Thus $c = \frac{1}{a_1 \dots a_m}$ and

$$p_{a_1,\dots,a_m}(t) \sim \frac{t^{m-1}}{a_1 a_2 \dots a_m (m-1)!}$$

as $x \to \infty$

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6. Fourier-Dedekind sums

In this section we give a reciprocity result for Fourier-Dedekind sums. In many ways, the Dedekind sums extend the notion of the greatest common divisor of two integers. Following surprising result is an instance of reciprocity of Fourier-Dedekind sums.

Theorem 6.1 (Zagier Reciprocity)**.** *For all relatively prime positive integers $a_1, a_2, \dots, a_d$,*

$$s_0(a_2, a_3, \dots, a_d; a_1) + s_0(a_1, a_3, a_4, \dots, a_d; a_2) + \dots + s_0(a_1, a_2, \dots, a_{d-1}; a_d)$$
$$= 1 - \mathrm{poly}_{a_1,a_2,\dots,a_d}(0)$$

Proof. We compute the constant term of quasipolynomial $p_A(n)$

$$p_A(0) = \text{poly}_{a_1,a_2,\ldots,a_d}(0) + s_0(a_2, a_3, \ldots, a_d; a_1) + s_0(a_1, a_3, a_4, \ldots, a_d; a_2) + \ldots$$
$$+ s_0(a_1, a_2, \ldots, a_{d-1}; a_d)$$

using the fact that Ehrhart quasipolynomial $L_{\mathcal{P}}$ of the rational convex polytope $\mathcal{P} \subset \mathbb{R}^d$ satisfies $L_{\mathcal{P}}(0) = 1$ we have

$$1 = \text{poly}_{a_1,a_2,\ldots,a_d}(0) + s_0(a_2, a_3, \ldots, a_d; a_1) + s_0(a_1, a_3, a_4, \ldots, a_d; a_2) + \ldots$$
$$+ s_0(a_1, a_2, \ldots, a_{d-1}; a_d)$$

as desired. ■

As a corollary we have

Corollary 6.2. *For pairwise relatively prime positive integers a, b, c*

$$s_0(a, b; c) + s_0(b, c; a) + s_0(c, a; b) = 1 - \frac{1}{12}\left(\frac{3}{a} + \frac{3}{b} + \frac{3}{c} + \frac{a}{bc} + \frac{b}{ac} + \frac{c}{ab}\right)$$

7. Alternate expression of restricted partition function

In this section we prove simplified expression for the restricted partition function which is the main result of the recent paper [2]. We adopt its proof from [3]

Theorem 7.1. *The restricted partition function may be expresed as*

$$p_{\boldsymbol{a}}(n) = \sum_{\substack{\boldsymbol{j} \in J \\ a_1 j_1 + \ldots + a_r j_r \cong n \pmod{D}}} \binom{\frac{n - a_1 j_1 + \ldots + a_r j_r}{D} + r - 1}{r - 1}$$

where $D = \text{lcm}(a_1, \ldots, a_r)$ and the summation index runs over

$$J = \left\{ \boldsymbol{j} = (j_1, \ldots, j_r) | \, \boldsymbol{j} \in \mathbb{N}^r, 0 \le j_1 \le \frac{D}{a_1} - 1, \ldots, 0 \le j_r \le \frac{D}{a_r} - 1 \right\}$$

Proof. From the generating function

$$\sum p_{\mathbf{a}}(n) z^n = \prod \frac{1}{1 - z^{a_i}}$$

can be transformed by forcing each term to have common denominator $1 - z^D$

$$\frac{1}{1 - z^{a_i}} = \frac{\frac{1 - z^D}{1 - z^{a_i}}}{1 - z^D} = \frac{\sum_{k=0}^{\frac{D}{a_i} - 1} z^{k a_i}}{1 - z^D}$$

which yields

$$\sum p_{\mathbf{a}}(n) z^n = \frac{1}{(1 - z^D)^r} \sum_{\mathbf{j} \in J} z^{a_1 j_1 + \ldots + a_r j_r}$$

Expanding the denominator

$$\frac{1}{(1 - z^D)^r} = \sum_{p=0}^{\infty} \binom{p + r - 1}{r - 1} z^{pD}$$

which gives

$$\sum p_{\mathbf{a}}(n)z^n = \sum_{\substack{\mathbf{j}\in J \\ p\geq 0}} \binom{p+r-1}{r-1} z^{pD+a_1j_1+\ldots+a_rj_r}$$

Identifying the coefficient of z^n in both expressions we have

$$p_{\mathbf{a}}(n) = \sum_{\mathbf{j}\in J} \binom{\frac{n-a_1j_1+\ldots+a_rj_r}{D}+r-1}{r-1}$$

where the summation consists of all such indices $\mathbf{j}$ such that

$$pD + a_1j_1 + \ldots + a_rj_r = n$$

which is equivalent to the condition

$$a_1j_1 + \ldots + a_rj_r \cong n \pmod D$$

■

References

[1] M. Beck and S. Robins, Computing the Continuous Discretely, Integer-Point Enumeration in Polyhedra, Springer, Second Edition, 2015.

[2] M. Cimpoeaş and F. Nicolae, On the restricted partition function, arXiv:1609.06090v6

[3] S. Robins and C. Vignat, Simple proofs and expressions for the restricted partition function and its polynomial part, arXiv:1802.07310

[4] J. Ball and D. Welsh, The Diophantine Frobenius Problem, Oxford Lecture Series in Mathematics and its Applications 30, 2005

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