PÓLYA'S ENUMERATION THEOREM

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Abstract. In this paper, we aim to explain Pólya's Enumeration Theorem and the details behind it. We will cover group theory/algebra, the Orbit-Stabilizer Theorem, Burnside's Lemma, and some other interesting theorems, and finally Pólya's Enumeration Theorem and its proof. Some knowledge of group theory is useful for understanding this paper's content, but there will be some review as we go along. The majority of this paper is based on Alec Zhang's work in his paper Pólya's Enumeration [Zha].

1. Review of Basic Group Theory

We will begin by briefly recalling some definitions in group theory.

Definition 1.1. A *group* is a set G with an operation $*$ which satisfies the following properties:

- Associativity: For all $a, b, c \in G$, $(a * b) * c = a * (b * c)$.
- Identity: There exists an *identity element* $e \in G$ such that, for all $a \in G$, $e * a = a * e = a$.
- Inverse: For all $a \in G$, there exists inverse $a^{-1} \in G$ such that $a * a^{-1} = a^{-1} * a = e$.

If we have two elements $g, h \in G$, we denote $g * h$ as simply gh. Additionally, a *subgroup* H of G, denoted $H \leq G$, is a group under the same operation of G whose elements are all contained in G.

One fundamental example of a group is the *symmetric group*, denoted S_n, whose elements are permutations of the set $\{1, \ldots, n\}$ with operation composition. Another important definition, which will be used in the very definition of Pólya's Enumeration Theorem:

Definition 1.2. For a group G and set X, a *left group action* is a function $\phi : G \times X \to X$ (i.e., from the direct product of sets G and X to X) which satisfies the following properties for left identity and compatibility:

- Left Identity: For the identity element $e \in G$, for all $x \in X$, $\phi(e, x) = x$.
- Left Compatibility: For all $g, h \in G$, for all $x \in X$, $\phi(gh, x) = \phi(g, \phi(h, x))$.

Similarly, the function ϕ must satisfy the following properties for right identity and compatibility:

- Right Identity: For the identity element $e \in G$, for all $x \in X$, $\phi(x, e) = x$.
- Right Compatibility: For all $g, h \in G$, for all $x \in X$, $\phi(x, gh) = \phi(\phi(x, g), h)$.

Note that, when not specified, a *group action* is generally taken to be a left group action, but throughout this paper, everything which applies to left group actions will also apply to right group actions. Furthermore, for a group element $g \in G$, set element $x \in X$, and group action ϕ, the left group action $\phi(g, x)$ is denoted gx, and the right group action $\phi(x, g)$ is denoted xg.

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From what we know about group actions, we have the following proposition:

Proposition 1.3. *Given a group action ϕ of group G on a set X, the function $f_\phi : x \mapsto \phi(g, x)$ (i.e., the function mapping element x to $\phi(g, x)$) is bijective for all $g \in G$.*

Now, let's go over some other definitions which we will need shortly to present the Orbit-Stabilizer Theorem and Burnside's Lemma. We begin by looking at the definition of an *orbit*.

Definition 1.4. Given a group action ϕ of a group G on a set X, the *orbit* of a set element $x \in X$ is

$$\text{orb}(x) = O_x = \{gx : g \in G\} = y \in X,$$

such that there exists an element $g \in G : y = gx$.

Since there is a lot of notation involved in this definition, another way to think about the orbit of $x \in X$ is as the set of elements in X which x can be mapped to by the elements of G. Next, let's look at the definition of a *stabilizer*.

Definition 1.5. Given a group action ϕ of a group G on a set X, the *stabilizer* of a set element $x \in X$ is

$$\text{stab}(x) = S_{xx} = \{g \in G | gx = x\}.$$

Put another way, the stabilizer is the set of elements $g \in G$ where the (left) group action is x. Now we can denote the *transformer* in a similar way to the stabilizer, except with an element y in place of one of the elements x.

Definition 1.6. Given a group action ϕ of a group G on a set X, the *transformer* of two set elements $x, y \in X$ is

$$\text{trans}(x, y) = S_{xy} = \{g \in G | gx = y\}.$$

Finally, let's look at the *quotient*, which we will use in the next proposition.

Definition 1.7. Given a group action ϕ of a group G on a set X, the *quotient* of ϕ is

$$X/G = \{O_x : x \in X\}.$$

Proposition 1.8. *For any group action ϕ of a group G on a set X, X/G is a partition of X.*

Proof. Because equivalence classes of a set partition it, it just needs to be shown that $x \sim y \iff x, y_x$ is an equivalence relation, i.e., a relation between elements of a set which satisfies the reflexive, symmetric, and transitive properties. These are not difficult to check:

- Reflexive Property: For all $x \in X$, $x \sim x$ because $ex = x \in O_x$ for the identity element $e \in G$.
- Symmetric Property: For all $x, y \in X$, $x \sim y \implies y \sim x$.
- Transitive Property: For all $x, y, z \in X$, if $x \sim y$ and $y \sim z$, then $x, y, z \in O_x$, so $x \sim z$.

□

Here is another proposition, which states that the stabilizer of any element $x \in X$ forms a subgroup:

Proposition 1.9. *For any group action ϕ of a group G on a set X, $S_{xx} \leq G$ for every $x \in X$.*

Proof. We need to check associativity, closure, identity, and inverse properties of G. Let us specify elements $g_i, g_j \in S_{xx}$, and for $x \in X$, we have the following:

- Associativity: This follows from the group structure.
- Closed: We have $g_i(g_j x) = g_i x = x$, as well as $(g_i g_j)x = x \implies g_i g_j \in S_{xx}$.
- Identity: The identity $e \in G$ is in S_{xx} because $ex = x$.
- Inverse: For some arbitrary $g_k \in S_{xx}$, we have $g_k x = x$, and $g_k^{-1}(g_k x) = g_k^{-1}x \implies g_k^{-1} = (g_k^{-1} g_k)x = ex = x$. This follows from compatibility of ϕ, so $g_k^{-1} \in S_{xx}$.

□

2. The Orbit-Stabilizer Theorem and Burnside's Lemma

Before we can look at Pólya's Enumeration Theorem, we must first learn about one lemma and two other important theorems relating to the group theory we've just encountered.

Lemma 2.1. *For all $y \in O_x$, $|S_{xx}| = |S_{xy}|$.*

Proof. First, let $g_{xx} \in S_{xx}$ and $g_{xy} \in S_{xy}$. Then, we have that $g_{xy} g_{xx} x = g_{xy} x = y$, therefore $g_{xy} g_{xx} \in S_{xy}$. Since $g_{xy} x = y$, when we multiply by g_{xy}^{-1} we get (by compatibility) $g_{xy}^{-1} g_{xy} x = g_{xy}^{-1} y = ex = x$, so $g_{xy}^{-1} \in S_{yx}$ and $g_{xy}^{-1} g_{xy} \in S_{xx}$. We know that $g_{xy} g_{xx} \in S_{xy}$, so define the function $a : S_{xx} \to S_{xy} : g_{xx} \mapsto h g_{xx}$. We know that $g_{xy}^{-1} g_{xy} \in S_{xx}$, so define the function $b : S_{xy} \to S_{xx} : g_{xy} \mapsto h^{-1} g_{xy}$. Because b is the inverse of a, we have that

$$a(b(g_{xy})) = a(h^{-1} g_{xy}) = h h^{-1} s_{xy} = g_{xy},$$
$$b(a(g_{xx})) = b(h g_{xx}) = h^{-1} h g_{xx} = g_{xx}.$$

Therefore, a is a bijection, and $|S_{xx}| = |S_{xy}|$. □

Theorem 2.2 (Orbit-Stabilizer Theorem). *Given any group action ϕ of a group G on a set X, for all $x \in X$,*

$$|G| = |S_{xx}||O_x|.$$

Now, let there be an element $h \in S_{xy}$.

Proof. By Lemma 2.1, we have that $|S_{xy}| = |S_{xx}|$ for every $y \in O_x$. Since the group action is a function, G must have set partitions $S_{xy} : y \in O_x$. To split G into these two partitions, this constitutes changing the y to an x in S_{xy}, because $y \in O_x$. Therefore, $|G| = |S_{xx}||O_x|$, and this completes the proof. □

Theorem 2.3 (Burnside's Lemma). *Given a finite group G, a finite set X, and a group action ϕ of G on X, the number of distinct orbits is*

$$|X/G| = \frac{1}{|G|} \sum_{g \in G} |X^g|,$$

where $X^g = \{x \in X | gx = x\}$, the set of elements of X fixed by action by g.

Proof. We begin by noting that

$$\sum_{g \in G} |X^g| = |(g, x) \in (G, X) : gx = x| = \sum_{x \in X} |S_{xx}|.$$

We would like to show that

$$|X/G| = \frac{1}{|G|}\sum_{x\in X}|S_{xx}|.$$

By the Orbit-Stabilizer Theorem, we have that

$$\frac{1}{|G|}\sum_{x\in X}|S_{xx}| = \frac{1}{|G|}\sum_{x\in X}\frac{|G|}{|O_x|} = \sum_{x\in X}\frac{1}{|O_x|},$$

because $|S_{xx}| = \frac{|G|}{|O_x|}$. Let's rewrite the sum, since we know that orbits partition X. We have

$$\sum_{x\in X}\frac{1}{|O_x|} = \sum_{A\in X/G}\sum_{x\in A}\frac{1}{|A|} = \sum_{A\in X/G} 1 = |X/G|.$$

This gives us $|X/G| = \frac{1}{|G|}\sum_{x\in X}|X^g|$. □

With these proofs completed, we now turn to the main topic of this paper, Pólya's Enumeration Theorem.

3. Pólya's Enumeration Theorem

Before we can understand the theorem, we will discuss some notation and a few more definitions. Firstly, let's look at the notation used in this section. We will be considering functions f from finite set X to finite set Y. We denote the set of all functions $f : X \to Y$ as Y^X, represented as a set of ordered pairs $(x_i, f(x_i))$ for $x_i \in X$. Note that an action ϕ of G on X gives us a natural group action[1] ϕ' of G on Y^X, where $f \in Y^X$:

$$\phi' : (g, f) \mapsto f' = f \circ p_g^{-1} = \{(\phi(g, x), f(x)) | x \in X\}.$$

It is not difficult to check that ϕ' also satisfies identity and compatibility. Now let's look at some definitions, some of which are review from graph theory:

Definition 3.1. Let p be a permutation of the elements in X. Then the *type* of p is the set $\{b_1, \ldots, b_n\}$, where each b_i is the number of cycles of length i in the cycle decomposition of p.

Definition 3.2. The *cycle index polynomial* Z_ϕ of the group action ϕ is defined as

$$Z_\phi(x_1, \ldots, x_n) = \frac{1}{|G|}\sum_{g\in G}\prod_{i=1}^{n} x_i^{b_i(g)},$$

where $b_i(g)$ is the i^{th} element of the type of the implied permutation $p_g \in \text{Sym}(X)$.

Definition 3.3. Two functions $f_1, f_2 \in Y^X$ are said to be *equivalent* under the action of (denoted as) $G(f_1 \sim_G f_2)$ if they are in the same orbit of ϕ', i.e., there exists $g \in G$ such that $f_2 = gf_1$.

Definition 3.4. A *configuration* is an equivalence class of the equivalence relation $\sim_G$ on Y^X. In other words, a configuration c is an orbit of ϕ', and the set C of configurations is Y^X/G under ϕ'.

[1]A *natural group action* is essentially the action $Sym(S) \times S \to S$ given by $(f, x) \mapsto f(x)$.

Definition 3.5. Let $w : Y \to \mathbb{R}$ denote the correspondence of a weight to each element in Y. Then, the *weight* of a function $f \in Y^X$ is

$$W(f) = \prod_{x \in X} w(f(x)).$$

At this point, we might be wondering if each function in configuration C has a common weight. The following proposition states that this is the case.

Proposition 3.6. *All functions in a configuration have the same common weight.*

Proof. Let us consider functions $f_1, f_2 \in Y^X$ in configuration C. Then there must be some $g \in G$ for which $f_1(gx) = f_2(x)$. This is because $f_1 \sim_G f_2$. Furthermore, we have that $\prod_{x \in X} w(f(x)) = \prod_{x \in X} w(f(gx))$ for any $g \in G$. Therefore,

$$W(f_1) = \prod_{x \in X} w(f_1(x)) = W(f_2).$$

□

We just need one more definition, and we will be prepared to look at the enumeration theorem.

Definition 3.7. Let C be the set of configurations c. The *configuration generating function (CGF)* is

$$F(C) = \sum_{c \in C} W(c).$$

Finally, here are the two versions of Pólya's Enumeration Theorem, one considered the unweighted version, and the other considered the weighted version.

Theorem 3.8 (Pólya's Enumeration Theorem (Unweighted))**.** *Let G be a group and let X and Y be finite sets, where $|X| = n$. Then, for any group action ϕ of G on X, the number of distinct configurations in Y^X is*

$$|C| = \frac{1}{|G|} \sum_{g \in G} |Y|^{c(g)},$$

where $c(g)$ is the number of cycles in the cycle decomposition of $p_g \in Sym(X)$, the permutation of X associated with the action of g on X.

Proof. We have that $|C| = |Y^X/G|$ under ϕ'. This is because orbits of ϕ' are configurations. We can easily apply Burnside's Lemma to Y^X, and we get

$$|Y^X/G| = \frac{1}{|G|} \sum_{g \in G} |(Y^X)^g|.$$

There are $|Y|$ choices of elements in Y for each of the $c(g)$ cycles in the cycle decomposition, so $|(Y^X)^g| = |Y|^{c(g)}$. Substituting this for $|(Y^X)^g|$ in the previous equation, we obtain the desired result. □

In order to state and prove the much more complicated weighted version of the theorem, we first look at a lemma:

Lemma 3.9. *We have that $|C| = \frac{1}{|G|} \sum_{g \in G} |\{f \in Y^X | f(gx) = f(x)\}|$ for all $x \in X$.*

Proof. Let ϕ'_R be the right group action on Y^X induced by ϕ, where $f \in Y^X$ and $g \in G$:

$$\phi'_R : (f, g) \mapsto f'_R = f \circ p_g = \{(x, f(\phi(g, x)))|x \in X\}.$$

Similarly to the proof of the unweighted Pólya's Enumeration Theorem, we can apply Burnside's Lemma to Y^X, which yields the desired result. □

Theorem 3.10 (Pólya's Enumeration Theorem (Weighted))**.** *Let G be a group and let X and Y be finite sets, where $|X| = n$. Let w be a weight function on Y. Then, for any group action ϕ of G on X, the configuration generating function is given by*

$$Z_\phi \left(\sum_{y \in Y} w(y), \sum_{y \in Y} w(y)^2, \ldots, \sum_{y \in Y} w(y)^n \right).$$

Proof. This proof is significantly more difficult than the proof of the unweighted version, and relies indirectly on Burnside's Lemma through the use of Lemma 3.9. We begin by taking ϕ'_R to be our group action on Y^X. Consider the set of all configurations c with common weight w, denoted $A(w) = \{c \in C|W(c) = w\}$. Consider also the set of all functions stabilized by g, denoted $S_{gg} = \{f \in Y^X|f = fg\}$. Then, let $S_{gg}(w) = \{f \in Y^X|f = fg, W(f) = w\}$ denote the set of all functions with stabilizer g with common weight w. By Lemma 3.9 and Burnside's Lemma, we have

$$|A(w)| = \frac{1}{|G|} \sum_{g \in G} |S_{gg}(w)|.$$

If we group the configuration generating function by weights, we have that this generating function is

$$\sum_{c \in C} W(c) = \sum_{w} w|A(w)| = \frac{1}{|G|} \sum_{w} \sum_{g \in G} w|S_{gg}(w)|,$$

and flipping the summations, we get

$$\frac{1}{|G|} \sum_{g \in C} \sum_{w} w|S_{gg}(w)| = \frac{1}{|G|} \sum_{g \in G} \sum_{f \in S_{gg}} W(f)$$

as the configuration generating function. Now, note that by the group action, X is permuted by G, therefore the permutation p_g for $g \in G$ has cycle decomposition $C_1, \ldots, C_k$ for $k \leq n$. This means that if $f \in S_{gg}$, then $f(x) = f(gx) = f(g^2x) = \ldots$, for all $x \in X$, $g \in G$, and f constant on each cycle C_i in the cycle decomposition. We have the following:

$$\sum_{f \in S_{gg}} W(f) = \sum_{f \in S_{gg}} \prod_{x \in X} w(f(x)) = \sum_{f \in S_{gg}} \prod_{i=1}^{k} \prod_{x \in C_i} w(f(x)) = \sum_{f \in S_{gg}} \prod_{i=1}^{k} w(f(x_i))^{|C_i|},$$

where $x_i \in C_i$. Now, let $|Y| = m$. Let us find a way to cover all possible assignments of $y \in Y$ to cycles C_i. We have

$$\sum_{f \in S_{gg}} W(f) = \prod_{i=1}^{k} \left(w(y_1)^{|C_i|} + \ldots + w(y_m) \right)^{|C_i|} = \prod_{i=1}^{k} \sum_{y \in Y} w(y)^{|C_i|}.$$

If we recall the expression for the configuration generating function which we found earlier, we can plug in the $\prod_{i=1}^{k}\sum_{y\in Y} w(y)^{|C_i|}$ which we just found:

$$\frac{1}{|G|}\sum_{g\in G}\left(\prod_{i=1}^{k}\sum_{y\in Y} w(y)^{|C_i|}\right).$$

By definition of type, there are $b_j(g)$ cycles of length j, and note that cycle length doesn't matter here. Therefore, we have that the configuration generating function is

$$\frac{1}{|G|}\sum_{g\in G}\prod_{j=1}^{n}\left(\sum_{y\in Y} w(y)^j\right)^{b_j(g)} = Z_\phi\left(\sum_{y\in Y} w(y), \sum_{y\in Y} w(y)^2, \ldots, \sum_{y\in Y} w(y)^n\right).$$

This completes the proof. □

References

[Zha] Alec Zhang. Polya's enumeration. *University of Chicago.*