1 Snake Oil

The Snake Oil Method is a powerful techinuqe for evaluating combinatorial sums. There are several steps involved:

- First identify the free variable, say n. Interpret the summand as a function of n, $f(n)$.
- Construct a generating function F from f.
- Extract the coefficients of the generating function to find a closed form for f.

Problem 1. Prove that

$$\sum_{k=0}^{\infty}\binom{k}{n-k}=F_{n+1}.$$

Proof. Suppose we let

$$f(n)=\sum_{k=0}^{\infty}\binom{k}{n-k}$$

and

$$\begin{aligned}F(x)&=\sum_{n=0}^{\infty}x^n f(n)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{k}{n-k}x^n\\&=\sum_{k=0}^{\infty}x^k\sum_{n=k}^{2k}\binom{k}{n-k}x^{n-k}=\sum_{k=0}^{\infty}x^k\sum_{n=0}^{k}\binom{k}{n}x^n\\&=\sum_{k=0}^{\infty}x^k(1+x)^k=\sum_{k=0}^{\infty}(x\cdot(x+1))^k\\&=\frac{1}{1-x-x^2}.\end{aligned}$$

That is, $F(x)$ is the generating function for the Fibonacci sequence. Hence, as desired, $f(n)=F_{n+1}$. ■

Problem 2. Prove that:

$$\sum_{k=0}^{\infty}\binom{n+k}{2k}2^{n-k}=\frac{1}{3}+\frac{2}{3}4^n.$$

Proof. Let

$$f(n)=\sum_{k=0}^{\infty}\binom{n+k}{2k}2^{n-k}$$

and

$$\begin{aligned}F(x) &= \sum_{n=0}^{\infty} x^n f(n) \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \binom{n+k}{2k} 2^{n-k} x^n \\ &= \sum_{k=0}^{\infty} x^k \sum_{n=k}^{\infty} \binom{n+k}{2k} (2x)^{n-k} \\ &= \sum_{k=0}^{\infty} x^k \sum_{n=0}^{\infty} \binom{n+2k}{2k} (2x)^{n}\end{aligned}$$

Now, we need to evaluate:

$$g(a) = \sum_{n=0}^{\infty} \binom{n+a}{a} x^n.$$

To do so, notice that:

$$\begin{aligned}g(a)\cdot(1-x) &= \sum_{n=0}^{\infty} \binom{a+n-1}{n} x^n \\ &= g(a-1).\end{aligned}$$

Since $g(0) = \frac{1}{1-x}$, it follows by induction that $g(a) = \frac{1}{(1-x)^{a+1}}$.
Returning to the main problem, we have that:

$$\begin{aligned}F(x) &= \sum_{k=0}^{\infty} x^k \sum_{n=0}^{\infty} \binom{n+2k}{2k} (2x)^{n} \\ &= \sum_{k=0}^{\infty} x^k \left(\frac{1}{1-2x}\right)^{2k+1} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{1-2x}\right)\left(\frac{x}{(1-2x)^2}\right)^{k} \\ &= \left(\frac{1}{1-2x}\right)\cdot\left(\frac{1}{1-\frac{x}{(1-2x)^2}}\right) \\ &= \frac{1-2x}{4x^2-5x+1} \\ &= \frac{1}{3}\cdot\frac{1}{1-x} + \frac{2}{3}\cdot\frac{1}{1-4x} \\ &= \sum_{n=0}^{\infty} \frac{1}{3}x^n + \frac{2}{3}(4x)^n,\end{aligned}$$

from which it follows that

$$f(n) = \frac{1}{3} + \frac{2}{3}\cdot 4^n,$$

as per the desired. ■

Problem 3. Show that:
$$\sum_{k=0}^{n}\binom{2n}{2k}\binom{2k}{k}2^{2n-2k}=\binom{4n}{2n}.$$

Proof. Let
$$f(n)=\sum_{k=0}^{n}\binom{2n}{2k}\binom{2k}{k}2^{2n-2k}.$$

Then:

$$\begin{aligned}F(x)&=\sum_{n=0}^{\infty}f(n)x^n\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{2n}{2k}\binom{2k}{k}2^{2n-2k}x^n\\&=\sum_{k=0}^{n}2^{-2k}\binom{2k}{k}\sum_{n=k}^{\infty}\binom{2n}{2k}\left(2\sqrt{x}\right)^{2n}.\end{aligned}$$

To evaluate the inner summand, we notice that:

$$\sum_{n=k}^{\infty}\binom{2n}{2k}\left(2\sqrt{x}\right)^{2n}=\frac{1}{2}\sum_{n=k}^{\infty}\binom{n}{k}\left(-2\sqrt{x}\right)^{n}+\frac{1}{2}\sum_{n=k}^{\infty}\binom{n}{k}\left(2\sqrt{x}\right)^{n},$$

which as we can recall from a previous problem evaluates to:

$$\frac{1}{2}\left(2\sqrt{x}\right)^{2k}\cdot\left(\frac{1}{\left(1-2\sqrt{x}\right)^{2k+1}}+\frac{1}{\left(1+2\sqrt{x}\right)^{2k+1}}\right).$$

Plugging this in, we see that:

$$\begin{aligned}F(x)&=\sum_{k=0}^{n}\frac{1}{2}\cdot\left(2\sqrt{x}\right)^{2k}\cdot 2^{-2k}\binom{2k}{k}\left(\frac{1}{\left(1-2\sqrt{x}\right)^{2k+1}}+\frac{1}{\left(1+2\sqrt{x}\right)^{2k+1}}\right)\\&=\frac{1}{\left(1-2\sqrt{x}\right)}\sum_{k}\binom{2k}{k}\left(\frac{x}{\left(1-2\sqrt{x}\right)}\right)^k+\frac{1}{\left(1+2\sqrt{x}\right)}\sum_{k}\binom{2k}{k}\left(\frac{x}{\left(1+2\sqrt{x}\right)}\right)^k\\&=\frac{1}{2(1-2\sqrt{x})}\cdot\frac{1}{\sqrt{1-\frac{4x}{(1-2\sqrt{x})^2}}}+\frac{1}{2(1+2\sqrt{x})}\cdot\frac{1}{\sqrt{1-\frac{4x}{(1+2\sqrt{x})^2}}}\\&=\frac{1}{2}\left(\frac{1}{\sqrt{1-4\sqrt{x}}}+\frac{1}{\sqrt{1+4\sqrt{x}}}\right).\end{aligned}$$

We'd like $F(x) = \sum \binom{4n}{2n} x^n$. To show that this is indeed the case, note that:

$$\begin{aligned}
G(x) &= \sum_{n=0}^{\infty} \binom{4n}{2n} x^n \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n}{n} x^n + \frac{1}{2} \sum_{n=0}^{\infty} \binom{2n}{n} (-x)^n \\
&= \frac{1}{2} \left(\frac{1}{\sqrt{1-4\sqrt{x}}} + \frac{1}{\sqrt{1+4\sqrt{x}}} \right) \\
&= F(x),
\end{aligned}$$

from which the desired follows. ■

Problem 4. Prove that:

$$\sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1} = \binom{n-1}{m-1}.$$

Proof. Let

$$f(n) = \sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1}.$$

We could also consider letting our free variable be m, but n seems more natural because it only appears once in the summand. As such:

$$\begin{aligned}
F(x) &= \sum_{n=0}^{\infty} f(n) x^n \\
&= \sum_n \sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1} x^n \\
&= \sum_k \binom{2k}{k} \frac{(-1)^k}{k+1} x^{-k} \sum_n \binom{n+k}{m+2k} x^{n+k}.
\end{aligned}$$

Seeking to evaluate the inner summand, let

$$\begin{aligned}
&g(x,m) = \sum_n \binom{n+k}{m+2k} x^{n+k} \\
&\implies xg(x,m) = \sum_n \binom{n+k-1}{m+2k} x^{n+k} \\
&\implies (1-x)g(x,m) = \sum_n \binom{n+k}{m+2k-1} x^{n+k+1} = xg(x,m-1).
\end{aligned}$$

And so, by induction, we see that:

$$g(x,m) = \frac{x^{m+2k}}{(1-x)^{m+2k+1}}.$$

Plugging this back in, we see that:

$$\begin{aligned} F(x) &= \sum_k \binom{2k}{k} \frac{(-1)^k}{(k+1)} x^{-k} \frac{x^{m+2k}}{(1-x)^{m+2k+1}} \\ &= \frac{x^m}{(1-x)^{m+1}} \sum_k \binom{2k}{k} \frac{1}{(k+1)} \frac{(-x)^k}{(1-x)^{2k}} \end{aligned}$$

To evaluate the inner summand, notice that

$$\begin{aligned} A(x) &= \sum_k \binom{2k}{k} \frac{1}{k+1} x^k \\ \implies xA'(x) &= \sum_k \binom{2k}{k} \frac{k}{k+1} x^k \\ \implies A(x) + xA'(x) &= \sum_k \binom{2k}{k} x^k \\ \implies A(x) + xA'(x) &= \frac{1}{\sqrt{1-4x}}. \end{aligned}$$

This is a first order differential equation, and it can be solved using elementary methods. With steps ommitted, we see that:

$$A(x) = \frac{\sqrt{1-4x}}{2x} - \frac{1}{2x}.$$

Hence,

$$\begin{aligned} F(x) &= \sum_k \frac{x^{m-1}}{2(1-x)^{m-1}} \cdot \left(1 - \sqrt{1 + \frac{4x}{(1-x)^2}}\right) \\ &= \frac{x^m}{(1-x)^m}. \end{aligned}$$

We notice that $F(x)$ is the generating function of the sequence $\binom{n-1}{m-1}$, from which the desired follows. ■