A Technique for Bivariate Main Diagonal Extraction

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1 Introduction

We are interested in a technique guaranteeing a useful extraction of the main diagonal of a bivariate generating function. That is, given a generating function $F(s,t) = \sum_{i,j\geq 0} f_{i,j} s^i t^j$, we wish to find a method to calculate

$$\text{diag } F = \sum_{n\geq 0} f_{n,n} x^n.$$

Such an extraction proves useful for counting lattice paths along the $y = x$ diagonal. We'll find that our proofs of the following theorem will conveniently provide a technique useful for the calculation of diag F as well as provide a useful condition on the simplicity of the result.

Theorem 1. *Let K be an algebraically closed field of characteristic zero, and let $F(s,t)$ be a rational function and a Laurent series over K. Then diag F is algebraic.*

The proof and technique presented here will forgo the use of complex analysis altogether and transform F into an appropriate Puiseux series. We will demonstrate the technique on two illustrative examples. The first will be the case of the number of North-East lattice paths along the plane diagonal, using steps of $(0,1)$ and $(1,0)$. The second will count the central Delannoy numbers, the number of lattice paths along the plane diagonal using steps of $(0,1),(1,0)$, and $(1,1)$.

2 Puiseux series proof

We first introduce some useful terminology.

Definition 1. *Let K be an algebraically closed field. Then $K[x]$ is the set of (finite) polynomials over K, $K[[x]]$ is the set of power series over K, and $K((x))$ is the set of Laurent series over K,* Note that in the context of lattice paths especially, we simply let $K = \mathbb{C}$.

Definition 2. *A fractional Laurent series or a Puiseux series in x is a series of the form*

$$\sum_{n\in\mathbb{Z}} a_n x^{n/D}$$

where $D \in \mathbb{Z}^+$. In other words, a Puiseux series generalizes a power series to include negative and fractional exponents with a bounded denominator. For a field K, we denote $K^{fra}((x))$ as the field of Puiseux series on x over K (i.e. whose coefficients a_n lie in K).

We now state Puiseux's theorem.

Theorem 2. *Puiseux's Theorem: Let K be an algebraically closed field of characteristic zero (such as $\mathbb{C}$). Then $K^{fra}((x))$ is algebraically closed.*

In other words, all solutions to a polynomial whose coefficients are Puiseux series over a certain field are themselves Puiseux series over the same field.

We can now prove **Theorem 1**.

Proof. Suppose $F(s,t)$ is a rational function represented by Laurent series $\sum_{n_s,n_t\in\mathbb{Z}} f_{m,n}s^m t^n$ such that $f_{m,n} \in K$ for all m and n. Let $G(x,s) = F(s, \frac{x}{s})$. We then have the exponent of $x = st$ marking a position along the diagonal and the exponent of s now marking the s-displacement away from the diagonal; thus, we see that $[s^0]G(x,s) = \operatorname{diag} F$.

Since F is a rational function in s and t, it follows that G is also rational and therefore can be expressed as $G = P/Q$ where P and Q are polynomials over K in s and x. We then factor $Q(x,s)$ as $k\prod_{i=1}^{l}(s-\xi_i(x))^{e_i}$ for some $k \in K$ with $e_i \in \mathbb{Z}^+$ for all $1 \le i \le l$. By Puiseux's theorem, we have

$$\xi_i(x) \in K((x^{1/r})) \subset K^{\text{fra}}((x))$$

for some $r \in \mathbb{Z}^+$. We then split G into partial fractions:

$$G(x,s) = \frac{P(x,s)}{Q(x,s)} = \frac{P(x,s)}{C\prod_{i=1}^{l}(s-\xi_i(x))^{e_i}} = \sum_{i=1}^{l}\frac{N_i(x,s)}{(s-\xi_i(x))^{e_i}}.$$

To transform G into a recognizable Laurent series in x and s, we partition $\xi_i(x)$ into two groups, $\alpha_i(x)$ for $1 \le i \le m$ and $\beta_j(x)$ for $1 \le j \le n$ such that the terms $\alpha_i(x)$ consist of series with strictly positive exponents and the terms $\beta_j(x)$ consist of the remaining roots that each have at least one non-positive exponent. Partitioning $\xi_i(x)$ thusly, we have

$$G(x,s) = \sum_{i=1}^{m}\frac{a_i(s)}{(s-\alpha_i(x))^{c_i}} + \sum_{j=1}^{n}\frac{b_j(s)}{(s-\beta_j(x))^{d_j}}$$

Consider the first summand. Note that $\alpha_i(x) \in K[[x^{1/r}]]$. The first summand can be rewritten as

$$\frac{A_i(s)}{(1-\alpha_i(x)s^{-1})^{c_i}}$$

where $A_i(s) = s^{c_i} a_i(s) \in K((x^{1/r}))[s^{-1}, s]$.

Consider the second summand. Note that while $\beta_j(x) \notin K[[x^{1/r}]]$, we have $\beta_j(x)^{-1} \in K[[x^{1/r}]$. The second summand can therefore be rewritten as

$$\frac{B_j(s)}{(1-s\beta_j(x)^{-1})^{d_i}}$$

where $B_j(s) = \beta_j^{-d_j} b_j(s) \in K((x^{1/r}))[s^{-1}, s]$.

Thus we have

$$G(x,s) = \sum_{i=0}^{m} \frac{A_i(s)}{(1-s^{-1}\alpha_i(x))^{c_i}} + \sum_{j=0}^{n} \frac{B_j(s)}{(1-s\beta_j(x)^{-1})^{d_j}}$$

and, expanding,

$$G(x,s) = \sum_{i=0}^{m} A_i(s) \sum_{k\geq 0} \binom{c_i}{k} (-1)^k s^{-k} \alpha_i(x)^k + \sum_{j=0}^{n} B_j(s) \sum_{k\geq 0} \binom{d_j}{k} (-1)^k s^k \beta_i(x)^{-k}$$

From this expansion we can see that since $A_i(s), B_j(s) \in K((x^{1/r}))[s, s^{-1}]$ and $\alpha_i(x), \beta_j(x)^{-1} \in K[[x^{1/r}]]$, we have

$$G(x,s) \in K[[x]]((s)) = K[[s, x/s]]$$

as necessary. Thus all manipulations have kept the expression in the appropriate ring.

We can now extract the diagonal as diag $F = [s^0]G(x,s)$, resulting in a sum of terms of the form $\gamma(x)\alpha(x)^a$ and $\delta(x)\beta(x)^{-b}$ where $a, b \in \mathbb{Z}^+$, $\gamma(x)$ and $\delta(x)$ are algebraic and $\alpha(x)$ and $\beta(x)^{-1}$ are algebraic. Thus diag F is algebraic. $\square$

3 Application of Puiseux series method

We now demonstrate the Puiseux series method for calculating the central diagonal on objects for which the diagonal appears naturally: two-dimensional plane lattice paths.

3.1 Puiseux series method for diagonal of North-East lattice paths

The number of North-East lattice paths to (i, j), consisting of steps $(0, 1)$ and $(1, 0)$, is generated by $F(s,t) = \frac{1}{1-s-t}$. We let $G(x,s) = F\left(s, \frac{x}{s}\right)$, obtaining

$$G(x,s) = \frac{1}{1-s-x/s} = \frac{-s}{s^2-s+x} = \frac{P(x,s)}{Q(x,s)}.$$

Factoring $Q(x,s)$ by s, we have $Q(x,s) = (s-\xi_-(x))(s-\xi_+(x))$ where $\xi_-(x) = \frac{1}{2}\left(1-\sqrt{1-4x}\right)$ and $\xi_+(x) = \frac{1}{2}\left(1+\sqrt{1-4x}\right)$.

By expanding, we see that ξ_- contains only positive powers of s and can therefore be appropriately relabeled α while ξ_+ contains a non-negative power of s and therefore can be appropriately relabeled β. Thus, expanding G into partial fractions accordingly, we have

$$G(x,s) = \frac{-s}{(s-\alpha)(s-\beta)} = \frac{\frac{\alpha}{\beta-\alpha}}{s-\alpha} - \frac{\frac{\beta}{\beta-\alpha}}{s-\beta} = \frac{\frac{\alpha}{\beta-\alpha}s^{-1}}{1-\alpha s^{-1}} + \frac{\frac{1}{\beta-\alpha}}{1-\beta^{-1}s}$$

and thus

$$\text{diag } F = [s^0]G(x,s) = \frac{1}{\beta-\alpha} = \frac{1}{\sqrt{1-4x}}.$$

Of course, this result is somewhat evident: we could easily note that

$$\text{diag } F = \sum_{n\geq 0} \binom{2n}{n} x^n,$$

which would lead to the same generating function. We proceed with a case for which direct calculation of the diagonal may be less obvious.

3.2 Puiseux series method for central Delannoy numbers

It can easily be found that number of Delannoy paths paths to (i,j), consisting of steps $(0,1),(1,0)$ and $(1,1)$ is generated by $F(s,t) = \frac{1}{1-s-t-st}$. The same approach can be applied. We include the computation for sake of illustration. As before, we let $G(x,s) = F\left(s, \frac{x}{s}\right)$,, obtaining

$$G(x,s) = \frac{1}{1-s-x/s-x} = \frac{-s}{s^2+s(x-1)+x} = \frac{P(x,s)}{Q(x,s)}.$$

Factoring $Q(x,s)$ by s, we similarly have $Q(x,s) = (s-\xi_-(x))(s-\xi_+(x))$ where $\xi_-(x) = \frac{1}{2}\left((1-x)-\sqrt{1-6x+x^2}\right)$ and $\xi_+(x) = \frac{1}{2}\left((1-x)+\sqrt{1-6x+x^2}\right)$.

By the same logic as before, we have

$$\text{diag } F = \frac{1}{\xi_+-\xi_-} = \frac{1}{\sqrt{1-6x+x^2}}.$$

4 References

Stanley P. Richard, *Enumerative Combinatorics: Volume 2*, Cambridge University Press, 1986.