THE WILF-ZEILBERGER METHOD

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Abstract.
The Wilf-Zeilberger method is a way to not only powerfully certify combinatorial identities, but also to discover new ones. In this paper we will go over the Wilf-Zeilberger algorithm as well as proofs of combinatorial identities that follow from this phenomenon.

1. An Introduction to Hypergeometric Functions

A hypergeometric function is a series $\mathcal{A}$ such that $a_{n+1} - a_n = \frac{t_{n+1}}{t_n}$ for some series $\mathcal{T}$. In other words, the difference between adjacent terms of the series can be written in terms of a ratio of consecutive terms of another series. Hypergeometric functions are very powerful in combinatorics, as we can easily find, certify, and prove combinatorial identities that can be written out in the form of a hypergeometric function. In particular, bivariate hypergeometric function identities can be certified using the WZ method.

2. What is a WZ Pair?

Assume we have a function $F(n, k)$ such that we wish to prove that $\sum_k F(n, k) = c$ for some c. Then if we could show that $\sum_k F(n+1, k) - F(n, k) = 0$, we would be able to prove this. Now, summing $F(n, k)$ and $F(n + 1, k)$ over k gives

$$\sum_k (F(n + 1, k) - F(n, k)) = 0.$$

Now, if only we could find a function G such that

$$G(n, k + 1) - G(n, k) = F(n + 1, k) - F(n, k),$$

we would have a telescoping series and could thus just work out

$$G(n, k \to \infty) - G(n, k \to -\infty),$$

and we would be done. We can even go so far as to find a function G such that $G(n, \pm\infty) = 0$, which would make our calculations even easier. This pair F, G is precisely a WZ pair.

Definition 2.1. Define a WZ pair to be a pair of functions $F(n, k)$ and $G(n, k)$ satisfying

$$F(n + 1, k) - F(n, k) = G(n, k + 1) - G(n, k)$$

with

$$\lim_{k\to\pm\infty} G(n, k) = 0 \text{ and } G = RF$$

for some rational function R [Her97].

This grammar is very useful in certifying combinatorial identities.

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3. WZ Pairs Prove Combinatorial Identities

Given the identity

$$\sum_k \binom{n}{k}^2 = \binom{2n}{n},$$

we can use the WZ method to provide a certificate of the identity.

Let

$$F(n,k) = \frac{\binom{n}{k}^2}{\binom{2n}{n}}.$$

Clearly if we can show that

$$\sum_k F(n,k) = 1,$$

then our identity must be true. Then we wish to find a function $G(n,k)$ such that F, G is a WZ pair. To find $\mathcal{G}$, we use Gosper's algorithm.

Definition 3.1. Define a hypergeometric function $\mathcal{Z}$ to be *Gosper-summable* if there exists $\mathcal{T}$ such that

$$t_n = z_{n+1} - z_n.$$

Gosper's algorithm recovers Z from a gosper-summable hypergeometric series $\mathcal{T}$, by expressing t_n in the form

$$t_n = \frac{a(n)}{b(n)} \cdot \frac{c(n+1)}{c(n)}$$

for $a(n), b(n), c(n)$ satisfying

$$\gcd(a(n), b(n+z)) = 1 \text{ for } z \geq 0.$$

The algorithm finds solutions $x(n)$ to the polynomial

$$a(n)x(n+1) - b(n-1)x(n) = c(n),$$

and returns the expression

$$\frac{b(n-1)x(n)}{c(n)t_n}.$$

Interestingly, we find that given $F(n,k)$, the $G(n,k)$ produced by Gosper's algorithm is always a rational function multiplied by $F(n,k)$. In other words, we have

$$G(n,k) = R(n,k)F(n,k). \text{ [Wil90]}$$

Thus if we find $R(n,k)$, our identity is true. We call this rational function a WZ certificate.

Application of Gosper's algorithm to our function

$$F(n,k) = \frac{\binom{n}{k}^2}{\binom{2n}{n}}$$

leads us to find

$$G(n,k) = F(n,k) \cdot \frac{k^2(2k-3n-3)}{2(k-n-1)^2(2n+1)}.$$

The existence of this hypergeometric series G proves the identity.

There are some combinatorial identities that follow very easily from the WZ method. An example of this is the WZ pair

$$F(n,k) = \frac{\binom{n}{k}}{2^n}, G(n,k) = -\frac{\binom{n}{k-1}}{2^{n+1}}.$$

A few algebraic manipulations show that

$$\begin{aligned} F(n+1,k) - F(n,k) &= 2^{-n-1}\binom{n+1}{k} - 2^{-n}\binom{n}{k} \\ &= 2^{-n-1}\left(\binom{n}{k} + \binom{n}{k-1}\right) - 2^{-n}\binom{n}{k} \\ &= 2^{-1}2^{-n}\left(\binom{n}{k} + \binom{n}{k-1}\right) - 2^{-n}\binom{n}{k} \\ &= 2^{-n-1}\binom{n}{k-1} - 2^{-1}\binom{n}{k}, \end{aligned}$$

which is precisely $G(n,k+1) - G(n,k)$. Thus, we have certified the identities

$$\sum_{k=0}^{\infty}\binom{n}{k} = 2^n$$

and

$$\sum_{k=0}^{\infty}\binom{n}{k-1} = 2^{n+1}$$

simultaneously.

Now, we can extend our definition of WZ pairs by using their properties to generate new ones. Once we have found one identity, it is easily possible to generate others.

4. An Extension of the WZ Method

Definition 4.1. Define the operation $D(\mathcal{WZ})$ on a WZ pair $\mathcal{WZ}$ to be the replacement of some number of occurrences of $(an+bk+c)!$ with $(-1)^{an+bk}/(-an-bk-c-1)!$. Note that $D(\mathcal{WZ})$ is a WZ pair, and this pair is called a dual pair of $\mathcal{WZ}$. We denote a dual pair by $\tilde{\mathcal{F}}, \tilde{\mathcal{G}}$.[Dor90]

We can use dual pairs to certify identities which follow from dual WZ pairs; examples include Vandermonde's identity, which can be used to certify the identity

$$\sum_k (-1)^{n+k}\binom{n}{k}\binom{k+a}{k} = \binom{a}{n}.$$

To see why this works, we note that

$$\binom{a}{k}\binom{n}{k} = \frac{a!n!}{k!(a-k)!k!(n-k)!}$$

and similarly

$$\binom{n+a}{a} = \frac{(n+a)!}{a!n!}.$$

Performing the operation $D(\mathcal{WZ})$ on this pair gives

$$(-1)^{n+k}\binom{n}{k}\binom{k+a}{k} = \binom{a}{n}$$

as desired.

Another example of a dual pair is the identity

$$\sum_k (-1)^{n+k}\binom{n}{k}2^k = 1[\text{Her97}],$$

which we can show is true as

$$\sum_k \binom{n}{k} = \sum_k \frac{n!}{k!(n-k)!} \implies \sum_k (-1)^{n-k}\frac{n!(-n+k-1)!}{k!} = 2^k,$$

and dividing this expression by our sum gives the result.

5. The Generalized Factorial

All of the combinatorial identities that we have proven so far involve infinite sums of binomial coefficients. Thus there must be a term for which $k > n$; letting $k = n + z$ and writing out our binomial coefficient as a product gives

$$\binom{n}{n+z} = \frac{n!}{(-z)!(n+z)!}.$$

We might start to wonder how we could possibly evaluate the factorial of a negative integer, and as a result of that, we introduce the generalized factorial.

Definition 5.1. Define the Gamma function to be

$$\Gamma(z) = \int_0^\infty z^{t-1}e^{-t}\,dt$$

for z with a positive real part.

Evaluating this integral using integration by parts gives

$$\begin{aligned}\Gamma(z) &= \int_0^\infty z^{t-1}e^{-t}\,dt \\ &= z^{t-1}e^{-t}\Big|_0^\infty + (z-1)\int_0^\infty z^{t-2}e^{-t}\,dt \\ &= (z-1)\int_0^\infty z^{t-2}e^{-t}\,dt \\ &= (z-1)\Gamma(z-1) \\ &= (z-1)!\Gamma(1).\end{aligned}$$

Noticing that $\Gamma(1) = 1$, we find

$$\Gamma(z) = (z-1)!.$$

This fact extends very nicely, as we have expanded the domain of the factorial operation to all complex numbers with a positive real part. However, we are interested in computing factorials of negative numbers, which we are still unable to do. Euler's reflection formula allows us to generalize the factorial operation even further.

Definition 5.2. Define Euler's reflection formula to be the identity

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(z\pi)}.$$

Applying this formula to the factorial representation of the gamma function gives us the result

$$(z-1)!(-z)! = \frac{\pi}{\sin(z\pi)} \implies (-z)! = \frac{\pi}{(z-1)!\sin(z\pi)}.$$

Using this generalization, we have extended the domain of the factorial operation to all complex numbers excluding negative integers. While we still cannot exactly compute the factorial of a negative integer due to division by zero, we can find the ratios between factorials of negative integers. We do this by taking the limit as $-z$ tends to an integer.

Definition 5.3. We can then formally define the generalized factorial

$$z! = \Gamma(z+1)$$

for $z \in \mathbb{C}$.

Using this fact we can generalize the binomial coefficients to any complex n and k. Revisiting our example from earlier, we find

$$\binom{n}{n+z} = \frac{n!}{(-z)!(n+z)!} = \frac{n!}{(n+z)!}\frac{(z-1)!\sin(z\pi)}{\pi} = 0.$$

This is very convenient, as it means that we can essentially ignore the terms for which $k > n$ in our sums.

Definition 5.4. Formally, we define the generalized binomial coefficients

$$\binom{n}{k} = \frac{\Gamma(n+1)}{\Gamma(k+1)\Gamma(n-k+1)}$$

for $n, k \in \mathbb{C}$.

With this generalized definition of the factorial and the binomial coefficients, we are ready to fully explore the WZ method.

6. Proofs of a Few Combinatorial Identities

Consider the identity

$$\sum_{k=0}^{\infty} \binom{n}{2k} = 2^{n-1}.$$

We can certify this identity using the WZ method, letting

$$F(n,k) = \frac{\binom{n}{2k}}{2^{n-1}}.$$

Then we find

$$G(n,k) = \binom{n}{2k} \frac{(2k-1)k}{2^{n-1}(2k-n-1)n},$$

and we can verify that $F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$.

We can also provide a certification of the identity

$$\sum_{k=0}^{\infty} k\binom{n}{k} = n \cdot 2^{n-1}$$

using the WZ method. First, note that

$$F(n,k) = \frac{k\binom{n}{k}}{n \cdot 2^{n-1}}.$$

To find $G(n,k)$, we apply Gosper's algorithm, which gives us

$$G(n,k) = \binom{n}{k} \frac{k(k-1)}{2^n n(k-n-1)}.$$

We find that

$$\begin{aligned} F(n+1,k) - F(n,k) - G(n,k+1) + G(n,k) &= \binom{n}{k}\frac{(k-1)k}{2^n(k-n-1)n} - \binom{n}{k}\frac{k}{n2^{n-1}} \\ &\quad + \binom{n+1}{k}\frac{k}{2^n(n+1)} - \binom{n}{k+1}\frac{(k+1)k}{(2^n(k-n)n)} \\ &= 0, \end{aligned}$$

and thus we have certified this identity as well.

What if we tried to find a dual identity of this? To do that, we first expand $F(n,k)$, which gives us

$$F(n,k) = \frac{(n-1)!}{(k-1)!(n-k)! \cdot 2^{n-1}}.$$

Performing the operation $D(\mathcal{F})$ gives us

$$\tilde{F}(n,k) = \frac{(k-n-1)!}{(k-1)!(-n)!2^{n-1}} = \frac{\binom{k-n-1}{k-1}}{2^{n-1}},$$

and using the WZ method, we generate

$$\tilde{G}(n,k) = \frac{(k-1)(k-2)\binom{k-n-1}{k-1}}{(k-n-1)(n-1)}$$

and we can verify that $\tilde{\mathcal{F}}, \tilde{\mathcal{G}}$ is a valid WZ pair.

Then $\sum_k \tilde{F}(n,k)$ is constant, so we find

$$\sum_{k=0}^{\infty} \tilde{F}(n,k) = \sum_{k=0}^{\infty} \frac{(k-n-1)!}{(k-1)!(-n)!2^{n-1}}$$
$$= \frac{1}{2^{n-1}} \sum_{k=0}^{\infty} \frac{(k-n-1)!}{(k-1)!(-n)!}$$

We can use the generalized factorial to simplify $\tilde{F}(n,k)$ to find

$$\tilde{F}(n,k) \to \frac{\Gamma(k-n-1)}{\Gamma(k-1)} \frac{(n-1)!\sin(z\pi))}{\pi}.$$

As $\sin n\pi = 0$ for $n \in \mathbb{Z}$, we find that

$$\sum_{k=0}^{\infty} \binom{k-n-1}{k-1} = 0$$

for positive n; a trivial result.

We might be inclined to try to find a less obvious identity; to do this, we can use the identity

$$\sum_{k=0}^{\infty} \binom{n}{k} = 2^n.$$

Letting $F(n,k) = \frac{\binom{n}{k}}{2^n}$ and simplifying gives us

$$F(n,k) = \frac{n!}{k!(n-k)!2^n}.$$

We showed earlier that

$$G(n,k) = -\frac{\binom{n}{k-1}}{2^{n+1}} = -\frac{n!}{(k-1)!(n-k+1)!2^{n+1}}.$$

From here we note that $G(-k-1,-n), F(-n-1,k)$ is a WZ pair for all WZ pairs $F(n,k), G(n,k)$ [Her97] so we can manipulate this identity further to find

$$\tilde{F}(n,k) = -\frac{(-k-1)!}{(-n-1)!(-k+n)!2^k} = -\binom{-k-1}{-n-1}2^{-k}.$$

We can use the gamma function to write this out as a ratio, which gives us

$$\begin{aligned}\tilde{F}(n,k) &\to -2^{-k}\frac{\pi}{k!\sin(k+1)\pi}\frac{n!\sin(n+1)\pi}{\pi}\frac{(k-n-1)!\sin(k-n)\pi}{\pi} \\ &\to -2^{-k}\frac{n!(k-n-1)!\sin(k-n)\pi}{\pi k!} \\ &= 0.\end{aligned}$$

Now, we write out the sum, which gives us

$$\begin{aligned}\sum_{k=0}^{\infty}\tilde{F}(n,k) &= \sum_{k=0}^{\infty}-\binom{-k-1}{-n-1}2^{-k} \\ &= -2^{-k}\sum_{k=0}^{\infty}\binom{-k-1}{-n-1} \\ &= \sum_{k=0}^{\infty}\binom{-k-1}{-n-1} = 0. \qquad (n=-1,0,1,2,3,...)\end{aligned}$$

Another simple result, yet elegant nonetheless.

7. Conclusion

Developing the grammar of WZ pairs, we went on to prove various combinatorial identities, as well as to explore some extensions of the Wilf-Zeilberger method which are used to find new identities that do not have a practical combinatorial proof associated with them. We finished our exploration of this method by finding some proofs of identities using the method, and we ended up finding a few simple identities.

References

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