Snake Oil Method

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1 Introduction

The snake oil method uses generating functions in order to evaluate sums. Instead of head-on trying to evaluate the sum, the method suggests we find the generating function for the generalized family of them and use the coefficients. While the method may be effective, there are some flaws to it that we should note before moving on. There aren't any flaws when applying the theorem to a solvable sum - assuming the mathematician doesn't make any errors -, but the Snake Oil method doesn't solve all sums. There are certain, hard sums that we cannot solve when applying the method. These sums, however, can be solved using computers. The Snake Oil method can be thought of in two different "categories"; one is used generally for easier sums and the other is used to solve binomial coefficient sums and other more difficult sums. In general, the method can be summed up with these steps:

1. Identify the variable, such as n, and name the sum as a function $f(n)$
2. Multiply the sum by x^n and sum over n
3. Change the order of the summations and perform the inner sum. In the background, we'll introduce expressions for certain sums that will help simplify the process
4. Identify the coefficients of the generating function

2 Background

There are a couple of series to know that will help simplify our evaluation of the upcoming examples. The most basic series that will help us is the binomial theorem:

$$\sum_r \binom{n}{r} x^r = (1+x)^r \tag{1}$$

Some other series that come up frequently when working with the snake oil method are mentioned in a paper by Professor Scharfenberger [Sch16]:

$$\sum_n \frac{1}{n+1}\binom{2n}{n} x^n = \frac{1}{2x}(1-\sqrt{1-4x}) \tag{2}$$

In addition to these two, it's also good to know the sums involving even and odd indices:

$$\sum_n \binom{m}{2n} x^{2n} = \frac{(1+x)^m + (1-x)^m}{2} \tag{3}$$

$$\sum_n \binom{m}{2n+1} x^{2n+1} = \frac{(1+x)^m - (1-x)^m}{2} \tag{4}$$

Some similar series - that are outlined by IMO Math as series that are helpful are [Mat]:

$$\sum_n \binom{2n}{m} x^{2n} = \frac{x^m}{2}\left(\frac{1}{(1-x)^{m+1}} + \frac{(-1)^m}{(1-x)^{m+1}}\right) \tag{5}$$

$$\sum_k \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}} \tag{6}$$

$$\sum_n \binom{2n+1}{m} x^{2n+1} = \frac{x^m}{2}\left(\frac{1}{(1-x)^{m+1}} - \frac{(-1)^m}{(1-x)^{m+1}}\right) \tag{7}$$

Finally, the following sum is also used frequently:

$$\sum_n \frac{1}{n+1}\binom{2n}{n} x^n = \frac{1}{2x}(1-\sqrt{1-4x}) \tag{8}$$

3 Snake Oil Theorem Applied

Let's start off with an example problem where we can apply the method:

Example 1.

$$\sum_{k\geq 0} \binom{k}{n-k}, \textit{ where } n \textit{ is a nonnegative integer.}$$

Proof. Our variable in play here is n, so we assign the function:

$$f(n) = \sum_{k\geq 0} \binom{k}{n-k}$$

Multiplying both sides by x^n and summing over n:

$$F(x) = \sum_n x^n \sum_{k\geq 0} \binom{k}{n-k}$$

Interchanging the sums:

$$F(x) = \sum_{k\geq 0} \sum_n \binom{k}{n-k} x^n$$

Now, we have to think of how to manipulate the expression inside the inner sum such that it looks like a sum that we're already familiar with. We can "split" x^n up such that:

$$F(x) = \sum_{k\geq 0} x^k \sum_n \binom{k}{n-k} x^{n-k}$$

Replacing $n-k$ with r gives us an expression that we're very familiar with:

$$F(x) = \sum_{k\geq 0} x^k \sum_n \binom{k}{r} x^r$$

The inner sum is the binomial theorem, which gives us:

$$\begin{aligned} F(x) &= \sum_{k\geq 0} x^k (1+x)^k \\ &= \sum_{k\geq 0} (x+x^2)^k \\ &= \frac{1}{1-x-x^2} \end{aligned} \tag{9}$$

We recognize this as the generating function for the Fibonacci numbers, so we have:

$$\sum_{k\geq 0} \binom{k}{n-k} = F_{n+1}$$

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Example 2.

$$\sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1}$$

Proof. We follow the same steps:

$$f(n) = \sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1}$$

$$F(x) = \sum_{n\geq 0} x^n \sum_k \binom{n+k}{m+2k} \binom{2k}{k} \frac{(-1)^k}{k+1}$$

$$F(x) = \sum_k \binom{2k}{k} \frac{(-1)^k}{k+1} x^{-k} \sum_{n\geq 0} \binom{n+k}{m+2k} x^{n+k}$$

$$F(x) = \sum_k \binom{2k}{k} \frac{(-1)^k}{k+1} x^{-k} \sum_{n\geq 0} \binom{r}{m+2k} x^r$$

By (2):

$$F(x) = \sum_k \binom{2k}{k} \frac{(-1)^k}{k+1} x^{-k} \frac{x^{m+2k}}{(1-x)^{m+2k}}$$

$$F(x) = \sum_k \binom{2k}{k} \frac{(-1)^k}{k+1} \cdot \frac{x^{m+k}}{(1-x)^{m+2k}}$$

$$F(x) = \frac{x^m}{(1-x)^{m+1}} \sum_k \binom{2k}{k} \frac{(-1)^k}{k+1} \left\{ \frac{-x}{(1-x)^2} \right\}^k$$

$$F(x) = \frac{x^m}{(1-x)^{m+1}} \sum_k \binom{2k}{k} \frac{(-1)^k}{k+1} \left\{ \frac{-x}{(1-x)^2} \right\}^k$$

Applying (9):

$$F(x) = \frac{x^m}{(1-x)^{m+1}} \left(\frac{1}{2\left[\frac{-x}{(1-x)^2}\right]} \left\{ 1 - \sqrt{1 - 4\left[\frac{-x}{(1-x)^2}\right]} \right\} \right)$$

$$F(x) = \frac{x^m}{(1-x)^{m+1}} \left(\frac{(1-x)^2}{-2x} \left\{ 1 - \sqrt{1 + \frac{4x}{(1-x)^2}} \right\} \right)$$

$$F(x) = \frac{-x^{m-1}}{2(1-x)^{m-1}} \left\{ 1 - \sqrt{1 + \frac{4x}{(1-x)^2}} \right\}$$

$$F(x) = \frac{-x^{m-1}}{2(1-x)^{m-1}} \left\{ 1 - \frac{1+x}{1-x} \right\}$$

$$F(x) = \frac{-x^{m-1}}{2(1-x)^{m-1}} \left(\frac{-2x}{1-x} \right)$$

$$F(x) = \frac{x^m}{(1-x)^m}$$

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Now for a slightly different example:

Example 3. Write the sum

$$f_n = \sum_{k \le \frac{n}{2}} (-1)^k \binom{n-k}{k} y^{n-2k}, \textit{ where } n \textit{ is non negative}$$

in simpler form.

Proof. First, we follow similar steps and try to find the ordinary generating function:

$$F(x) = \sum_{n \ge 0} x^n \sum_{2k \le n} (-1)^k \binom{n-k}{k} y^{n-2k}$$

$$F(x) = \sum_k (-1)^k y^{-2k} \sum_{n-2k \ge 0} \binom{n-k}{k} x^n y^n$$

$$F(x) = \sum_k (-1)^k y^{-k} x^k \sum_{n-k \ge k} \binom{n-k}{k} x^{n-k} y^{n-k}$$

$$F(x) = \sum_k (-1)^k y^{-k} x^k \sum_{r \ge k} \binom{r}{k} x^r y^r$$

By (2):

$$F(x) = \sum_{k\geq 0} (-1)^k y^{-k} x^k \frac{(xy)^k}{(1-xy)^{k+1}}$$

$$F(x) = \sum_{k\geq 0} (-1)^k \frac{x^{2k}}{(1-xy)^{k+1}}$$

$$F(x) = \sum_{k\geq 0} \frac{1}{1-xy} \left\{ \frac{-x^2}{(1-xy)} \right\}^k$$

We recognize the bracket portion to be a geometric series:

$$F(x) = \frac{1}{1-xy} \sum_{k\geq 0} \left\{ \frac{-x^2}{(1-xy)} \right\}^k$$

$$F(x) = \frac{1}{1-xy} \cdot \frac{1}{1-\left(\dfrac{-x^2}{1-xy}\right)}$$

$$F(x) = \frac{1}{1-xy} \cdot \frac{1-xy}{1-xy+x^2}$$

$$F(x) = \frac{1}{1-xy+x^2}$$

Now, to find the sum in closed form, we expand $F(x)$ using partial fractions:

$$F(x) = \frac{1}{(1-xx_+)(1-xx_-)} \tag{10}$$

$$F(x) = \frac{x_+}{(x_+ - x_-)(1-xx_+)} - \frac{x_-}{(x_+ - x_-)(1-xx_-)}$$

where

$$x_\pm = \frac{y \pm \sqrt{y^2-4}}{2}$$

Through some (pretty uninteresting) algebra, we get:

$$f_n = \frac{1}{\sqrt{y^2-4}} \left\{ \left(\frac{y+\sqrt{y^2-4}}{2} \right)^{n+1} - \left(\frac{y-\sqrt{y^2-4}}{2} \right)^{n+1} \right\}$$

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Example 4

Simplify

$$\sum_k \binom{n}{k} \binom{2k}{k} y^k$$

Proof. As usual we multiply by x^n and sum over n, but this time, we end up with a bivariate generating function:

$$F(x,y) = \sum_{n\geq 0} x^n \sum_k \binom{2k}{k}\binom{n}{k} y^k$$

$$F(x,y) = \sum_k \binom{2k}{k} y^k \sum_{n\geq 0} \binom{n}{k} x^n$$

$$F(x,y) = \sum_k \binom{2k}{k} y^k \frac{x^k}{(1-x)^{k+1}}$$

$$F(x,y) = \frac{1}{1-x} \sum_k \binom{2k}{k} \left(\frac{xy}{1-x}\right)^k$$

$$F(x,y) = \frac{1}{1-x} \sum_k \binom{2k}{k} \left(\frac{xy}{1-x}\right)^k$$

By (7) we have:

$$F(x,y) = \frac{1}{1-x} \left\{ \frac{1}{\sqrt{1-\frac{4xy}{1-x}}} \right\}$$

$$F(x,y) = \frac{1}{1-x} \left\{ \frac{1}{\sqrt{\frac{1-x-4xy}{1-x}}} \right\}$$

$$F(x,y) = \frac{1}{\sqrt{(1-x)(1-x(1+4y))}}$$

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Example 5

Let's say that we want to show that two sums are equal. For example, we want to prove that the following two sums are equal without evaluating either of the sums:

$$\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k} 2^k$$

Proof. Starting from the left:

$$\sum_{n\geq 0} x^n \sum_k \binom{m}{k}\binom{n+k}{m}$$

$$\sum_k \binom{m}{k} \sum_k \binom{n+k}{m} x^n$$

$$\sum_k \binom{m}{k} x^{-k} \sum_k \binom{n+k}{m} x^{n+k}$$

$$\sum_k \binom{m}{k} x^{-k} \sum_k \binom{r}{m} x^r$$

$$\sum_k \binom{m}{k} x^{-k} \frac{x^m}{(1-x)^{m+1}}$$

$$\left(1+\frac{1}{x}\right)^m \frac{x^m}{(1-x)^{m+1}}$$

$$\frac{(1+x)^m}{(1-x)^{m+1}}$$

On the right side:

$$\sum_{n\geq 0} x^n \sum_k \binom{m}{k}\binom{n}{k} 2^k$$

$$\sum_k \binom{m}{k} 2^k \sum_{n\geq 0} \binom{n}{k} x^n$$

$$\sum_k \binom{m}{k} 2^k \frac{x^k}{(1-x)^{k+1}}$$

$$\frac{1}{1-x} \sum_k \binom{m}{k} \left\{\frac{2x}{(1-x)}\right\}^k$$

$$\frac{1}{1-x}\left(1+\frac{2x}{1-x}\right)^m$$

$$\frac{(1+x)^m}{(1-x)^{m+1}}$$

We end up with the same generating function, and we've proved that the two sums are equal without even evaluating their actual values.

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This last example in particular shows the strength of the Snake Oil method when applied to complicated sums.

References

[Mat] IMO Math. The method of snake oil. https://www.imomath.com/index.php?options=357lmm=0. Accessed: 2021-11-01.

[Sch16] Jonas Scharfenberger. The snake oil method, 2016.