SNAKE OIL METHOD

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Abstract. In this paper, we introduce the Snake Oil method which is a powerful way to evaluate sums using generating functions. This method is very useful for handling a large variety of sums and combinatorial identities, as we will see.

Introduction

The snake oil method is powerful way to evaluate sums by turning them into generating function double sums, evaluating those, and extracting coefficients

Suppose we have a sum of the form

$$a_n = \sum_{k \geq 0} F(k, n)$$

for some function F where k is the variable being summed over and n is a free variable that is not. To evaluate this, we define

$$A(x) = \sum_{n \geq 0} a_n x^n = \sum_{n \geq 0} \left(\sum_{k \geq 0} F(k, n) \right) x^n = \sum_{n \geq 0} \sum_{k \geq 0} F(k, n) x^n.$$

We swap the order of summation to get

$$A(x) = \sum_{k \geq 0} \sum_{n \geq 0} F(k, n) x^n$$

and evaluate this instead.

Some Common Identities

Here we list some identities that we will use throughout the examples.

$$(1 + x)^n = \sum_{k \geq 0} \binom{n}{k} x^k.$$

$$\frac{x^m}{(1 - x)^{m+1}} = \sum_{k \geq 0} \binom{k}{m} x^k$$

$$\frac{1}{(1 - x)^{m+1}} = \sum_{k \geq 0} \binom{k + m}{m} x^k$$

$$\frac{1}{\sqrt{1 - 4x}} = \sum_{k \geq 0} \binom{2k}{k} x^k$$

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We also list a few identities only involving either even or odd indices. These are derived by adding/subtracting the above identities.

$$\sum_{k\geq 0}\binom{n}{2k}x^{2k}=\frac{(1+x)^n+(1-x)^n}{2}.$$

$$\sum_{k\geq 0}\binom{n}{2k}x^{2k}=\frac{(1+x)^n-(1-x)^n}{2}.$$

$$\sum_{k\geq 0}\binom{2k}{n}x^{2k}=\frac{x^n}{2}\left(\frac{1}{(1-x)^{n+1}}+\frac{(-1)^n}{(1-x)^{n+1}}\right).$$

$$\sum_{k\geq 0}\binom{2k+1}{n}x^{2k+1}=\frac{x^n}{2}\left(\frac{1}{(1-x)^{n+1}}-\frac{(-1)^n}{(1-x)^{n+1}}\right).$$

Examples

We start with the following example:

Example. Find

$$\sum_{k\geq 0}\binom{k}{n-k}.$$

Here n is the free variable and k is the one being summer over. We let $F(k,n)=\binom{k}{n-k}$. We define $A(x)=\sum_{n\geq 0}\sum_{k\geq 0}F(k,n)x^n$. Now swapping the order of summation, we get $\sum_{k\geq 0}\sum_{n\geq 0}F(k,n)x^n=\sum_{k\geq 0}\sum_{n\geq 0}\binom{k}{n-k}x^n$. We want this to look like something we know how to evaluate, and so we can write this as

$$\sum_{k\geq 0}x^k\sum_{n\geq 0}\binom{k}{n-k}x^{n-k}.$$

The inner sum is $(1+x)^k$, and therefore the entire sum becomes $\frac{1}{1-x-x^2}$. This is the generating function for the Fibonacci Numbers, and so $\sum_{k\geq 0}\binom{k}{n-k}=F_{n+1}$.

Example. Find

$$\sum_{k=m}^{n}\binom{n}{k}\binom{k}{m}.$$

What exactly is $F(k,n)$ here? Note that if n is fixed, then our sum depends on m which implies that m is a free variable. So here, we instead have $F(k,m)=\binom{n}{k}\binom{k}{m}$. So

$$A(x)=\sum_{m\geq 0}\sum_{k=m}^{n}\binom{n}{k}\binom{k}{m}x^m.$$

We swap the order of summation and write

$$\sum_{k\leq n}\binom{n}{k}\sum_{m\leq k}\binom{k}{m}x^m.$$

We evaluate the inner sum as $(1+x)^k$. So the expression becomes

$$\sum_{k\le n}\binom{n}{k}(1+x)^k.$$

This is $(1+(1+x))^n=(2+x)^n$.

Now we extract coefficients by expanding

$$(2+x)^n=\binom{n}{0}2^n+\binom{n}{1}2^{n-1}x+\cdots+\binom{n}{n-1}2x^{n-1}+\binom{n}{n}x^n.$$

Therefore, our desired sum is $\binom{n}{m}2^{n-m}$.

Now let's look at another example.

Example. Find

$$\sum_{k\ge 0}\binom{n+k}{2k}2^{n-k}.$$

We let $F(k,n)=\binom{n+k}{2k}2^{n-k}$ and define $A(x)=\sum_{n\ge 0}\sum_{k\ge 0}F(k,n)x^n$. Again, as we did earlier, we swap the order of summation and get

$$\sum_{k\ge 0}\sum_{n\ge 0}\binom{n+k}{2k}2^{n-k}x^n.$$

We want to write the $2^{n-k}x^n$ term as something nicer. We can either group out the 2^{-k} or the x^k term, both of which work. We group out the x^k term and get

$$\sum_{k\ge 0}x^k\sum_{n\ge 0}\binom{n+k}{2k}(2x)^{n-k}.$$

Then we see that this is

$$\sum_{k\ge 0}x^k\sum_{n\ge 0}\binom{(n-k)+2k}{2k}(2x)^{n-k}.$$

The inner series is something we know how to evaluate. Thus, we simplify the sum as

$$\sum_{k\ge 0}x^k\left(\frac{1}{1-2x}\right)^{2k+1}.$$

We write this as

$$\frac{1}{1-2x}\sum_{k\ge 0}\left(\frac{x}{(1-2x)^2}\right)^k.$$

This sum is a geometric series which we can write as

$$\frac{1}{1-2x}\frac{1}{1-\frac{x}{(1-2x)^2}}.$$

We simplify this:

$$\frac{1}{1-2x}\frac{1}{1-\frac{x}{(1-2x)^2}}=\frac{1-2x}{(1-2x)^2-x}=\frac{1-2x}{1-5x+4x^2}=\frac{1-2x}{(1-4x)(1-x)}=\frac{2/3}{1-4x}+\frac{1/3}{1-x}.$$

This gives the desired sum as

$$\frac{2^{2n+2}+1}{3}$$

Next, we see another use of the Snake Oil method. Suppose we have two fairly complicated sums, and we want to show that they are equal. One way we can do this is by showing that their generating functions are equal. Let's see an example of this:

Example. Show that

$$\sum_k \binom{m}{k}\binom{n+k}{m} = \sum_k \binom{m}{k}\binom{n}{k}2^k.$$

We first look at the first sum. We have

$$A(x) = \sum_n \sum_k \binom{m}{k}\binom{n+k}{m}x^n = \sum_k \sum_n \binom{m}{k}\binom{n+k}{m}x^n.$$

We take out an x^{-k} term to get

$$\sum_k \binom{m}{k}x^{-k}\sum_n \binom{n+k}{m}x^{n+k}.$$

We know that the inner sum is $\frac{x^m}{(1-x)^{m+1}}$. Therefore, we get

$$\sum_k \binom{m}{k}x^{-k}\frac{x^m}{(1-x)^{m+1}} = \frac{x^m}{(1-x)^{m+1}}\sum_k \binom{m}{k}x^{-k}.$$

This sum is $\left(1+\frac{1}{x}\right)^m$. So we get

$$A(x) = \frac{x^m}{(1-x)^{m+1}}\left(1+\frac{1}{x}\right)^m = \frac{(1+x)^m}{(1-x)^{m+1}}.$$

Now we look at the second sum. Here, we have

$$A(x) = \sum_n \sum_k \binom{m}{k}\binom{n}{k}2^k x^n = \sum_k \sum_n \binom{m}{k}\binom{n}{k}2^k x^n.$$

We write this as

$$\sum_k \binom{m}{k}2^k \sum_n \binom{n}{k}x^n.$$

We know that the inner sum is $\frac{x^k}{(1-x)^{k+1}}$. So we get

$$\sum_k \binom{m}{k}2^k\frac{x^k}{(1-x)^{k+1}} = \sum_k \binom{m}{k}\frac{(2x)^k}{(1-x)^{k+1}}.$$

We take out a factor of $\frac{1}{1-x}$ and get

$$\frac{1}{1-x}\sum_k \binom{m}{k}\left(\frac{2x}{1-x}\right)^k = \frac{1}{1-x}\left(1+\frac{2x}{1-x}\right)^m = \frac{(1+x)^m}{(1-x)^{m+1}}.$$

We see that the two generating functions are equal and thus, the sums must be equal too.

Let's see another example of this.

Example. Show that

$$\sum_k \binom{2n+1}{k}\binom{m+k}{2n} = \binom{2m+1}{2n}.$$

We look at the left sum. Here, we take m as our free variable. The generating function for this is

$$A(x) = \sum_m \sum_k \binom{2n+1}{k}\binom{m+k}{2n} x^m = \sum_m \sum_k \binom{2n+1}{k}\binom{m+k}{2n} x^m.$$

We take out the $\binom{2n+1}{k}$ and rewrite as

$$\sum_k \binom{2n+1}{2k} \sum_m \binom{m+k}{2n} x^m = \sum_k x^{-k} \sum_m \binom{m+k}{2n} x^{m+k}.$$

We know that the inner sum evaluates to $\frac{x^{2n}}{(1-x)^{2n+1}}$. Therefore, we have

$$\sum_k \binom{2n+1}{2k} x^{-k} \frac{x^{2n}}{(1-x)^{2n+1}} = \frac{x^{2n}}{(1-x)^{2n+1}} \sum_k \binom{2n+1}{2k} x^{-k}.$$

To make this nicer, we write

$$\frac{x^{2n}}{(1-x)^{2n+1}} \sum_k \binom{2n+1}{2k} \left(x^{-1/2}\right)^{2k}.$$

We know that

$$\sum_k \binom{2n+1}{2k} \left(x^{-1/2}\right)^{2k} = \frac{1}{2}\left(\left(1+\frac{1}{\sqrt{x}}\right)^{2n+1} + \left(1-\frac{1}{\sqrt{x}}\right)^{2n+1}\right).$$

Therefore,

$$A(x) = \frac{1}{2}(\sqrt{x})^{2n-1}\left(\frac{1}{(1-\sqrt{x})^{2n+1}} - \frac{1}{(1+\sqrt{x})^{2n+1}}\right).$$

Now we look at the right hand side of the given example. The generating function for this is

$$A(x) = \sum_m \binom{2m+1}{2n} x^m = x^{-1/2} \sum_m \binom{2m+1}{2n} \left(x^{1/2}\right)^{2m+1}.$$

This implies that

$$A(x) = x^{-1/2}\left[\frac{\left(x^{1/2}\right)^{2n}}{2}\left(\frac{1}{\left(1-x^{1/2}\right)^{2n+1}} - (-1)^{2n}\frac{1}{\left(1+x^{1/2}\right)^{2n+1}}\right)\right],$$

which we can rewrite as

$$A(x) = \frac{1}{2}(\sqrt{x})^{2n-1}\left(\frac{1}{(1-\sqrt{x})^{2n+1}} - \frac{1}{(1+\sqrt{x})^{2n+1}}\right).$$

We see that the generating functions are equal, and thus we are done.

Finally, we see a harder example where the Snake Oil method doesn't directly work.

Example. For given n and p, find

$$\sum_k \binom{2n+1}{2p+2k+1}\binom{p+k}{k}.$$

We assume that n is our free variable. For convenience, we take $r = p + k$. Then the desired sum is

$$\sum_r \binom{2n+1}{2r+1}\binom{r}{p}.$$

Since the binomial coefficient contains the term $2n + 1$, it turns out to be easier to take $A(x) = \sum_n a_n x^{2n+1}$ We have

$$A(x) = \sum_n \sum_r \binom{2n+1}{2r+1}\binom{r}{p} x^{2n+1} = \sum_r \sum_n \binom{2n+1}{2r+1}\binom{r}{p} x^{2n+1}.$$

We write this as

$$\sum_r \binom{r}{p} \sum_n \binom{2n+1}{2r+1} x^{2n+1}.$$

Now we see that

$$\sum_n \binom{2n+1}{2r+1} x^{2n+1} = \frac{x^{2r+1}}{2}\left(\frac{1}{(1-x)^{2r+2}} + \frac{1}{(1+x)^{2r+2}}\right).$$

Therefore we get

$$A(x) = \frac{1}{2}\frac{x}{(1-x)^2}\sum_r \binom{r}{p}\left(\frac{x^2}{(1-x)^2}\right)^r + \frac{1}{2}\frac{x}{(1+x)^2}\sum_r \binom{r}{p}\left(\frac{x^2}{(1+x)^2}\right)^r.$$

We write this as

$$A(x) = \frac{1}{2}\frac{x}{(1-x)^2}\frac{\left(\frac{x^2}{(1-x)^2}\right)^p}{\left(1-\frac{x^2}{(1-x)^2}\right)^{p+1}} + \frac{1}{2}\frac{x}{(1+x)^2}\frac{\left(\frac{x^2}{(1+x)^2}\right)^p}{\left(1-\frac{x^2}{(1+x)^2}\right)^{p+1}}.$$

We have

$$A(x) = \frac{1}{2}\frac{x^{2p+1}}{(1-2x)^{p+1}} + \frac{1}{2}\frac{x^{2p+1}}{(1+2x)^{p+1}} = \frac{x^{2p+1}}{2}\left((1+2x)^{-p-1} + (1-2x)^{-p-1}\right).$$

After some algebra, we find our desired sum to be

$$\binom{2n-p}{2n-2p} 2^{2n-2p}.$$

Lastly, we consider the following example:

Example. Prove that

$$\sum_i \binom{n}{i}\binom{2n}{i} = \binom{3n}{n}.$$

Note that this doesn't seem to work by our usual method since there are many occurrences of the free variable. However, we can generalize the above as

$$\sum_i \binom{n}{i}\binom{m}{r-i}$$

allowing more free variables, which we can then use the Snake Oil Method on.

References

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