# THE SNAKE OIL METHOD

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Abstract. This article discusses the snake oil method, an application of generating function theory to a wide range of combinatorial sums. We will go over plenty of examples of using the snake oil method, and discover some interesting identities along the way.

## 1. Introduction

We will begin by defining and giving important information about the snake oil method. First, the snake oil method is an *external approach* to solving combinatorial sum. The traditional *internal approach* would be to use combinatorial identities to rewrite the summand and create a more manageable sum, while the *external approach* would be to treat the summand as a function $f(n)$, solve for the generating function of $f(n)$, $F(z)$, and extract coefficients to get a closed form for $f(n)$. And now we give a more detailed look into the snake oil method:

**Algorithm 1.1.** *Snake Oil Method*
(1) Identify the free variable of the sum, let's call it $n$, that is not being summed over but that the summand depends on. We call the sum $f(n)$.
(2) We take $F(z)$ to be the ordinary generating function of $f(n)$, where $[z^n]F(z) = f(n)$.
(3) Since $F(z)$ is a double sum, we can interchange the order of the summations, and solve for the closed form of the inner summation.
(4) We extract coefficients from our generating function which should now be a single summation, this will give us a closed form for $f(n)$.

This method may seem a bit complicated, but after a few example one will see that it is quite straightforward and doesn't require as much thinking as other methods. We will note that the snake oil method cannot solve all combinatorial sums, but it has a very high success rate, and covers a wide variety of sums. Sums too hard for the snake oil method to compute are usually solved by computers using the rational function method, which will not be discussed in this article.

## 2. Important Generating Functions

Before covering numerous examples of the snake oil method being used, we cover some important sums which may be important later on. First,

$$\sum_{k\geq 0}\binom{n}{k}z^k = (1+z)^n$$

which is Newton's generalized binomial theorem. Next, we have the not as well known sum,

$$\sum_{n\geq 0}\binom{n}{k}z^n = \frac{z^k}{(1-z)^{k+1}}$$

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We will prove this using bivariate generating functions.

*Proof.* Consider the bivariate generating function,

$$F(z,u) = \sum_{n,k} \binom{n}{k} z^n u^k$$

We create the double sum,

$$F(z,u) = \sum_{n\geq 0} z^n \sum_{k\geq 0} \binom{n}{k} u^k$$

$$= \sum_{n\geq 0} z^n (1+u)^n = \sum_{n\geq 0} (z+zu)^n = \frac{1}{1-z(1+u)}$$

If we solve for $[u^k]F(z,u)$, we get

$$[u^k]F(z,u) = \frac{z^k}{(1-z)^{k+1}}$$

$$= \sum_{n\geq 0} \binom{n}{k} z^n$$

which proves the statement. ∎

Next we will consider the shifted fibonacci generating function,

$$F(z) = \sum_{k\geq 0} f_{k+1} z^k = \frac{1}{1-z-z^2}$$

The proof goes as follows:

*Proof.* First, we prove that the shifted fibonacci sequence counts compositions with parts of size 1 and 2. Every such composition of size $n$ is either composed of a composition of size $n-1$ with a 1 part at the end, or a composition of size $n-2$ with a 2 part at the end. If we call the number of such compositions of size $n$, $b_n$, we can see from the previous statement $b_{n+2} = b_{n+1} + b_n$. We can also compute $b_1 = 1$, $b_2 = 2$. These base cases and the recurrence are identical to those of the shifted fibonacci sequence, and so $f_{n+1} = b_n$. The generating functions must be the same, and the generating function for our compositions has specification $\mathrm{SEQ}(Z + Z \times Z) \to B(z) = \frac{1}{1-z-z^2}$ and this proves the statement. ∎

Next we find the exponential generating function for stirling numbers of the first kind,

$$C(z) = \sum_{k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix} \frac{z^n}{n!} = \frac{\left(\log \frac{1}{1-z}\right)^k}{k!}$$

*Proof.* Let us take $\mathcal{C}$ to be the combinatorial class of permutations with $k$ cycles. We have the specification $\mathcal{C} = \mathrm{SET}_k(\mathrm{CYC}(\mathrm{Z})) \to C(z) = \frac{\left(\log \frac{1}{1-z}\right)^k}{k!}$ which completes the proof. ∎

The final important generating function is the Bernoulli exponential generating function, which is

$$B(z) = \sum_{k\geq 0} \frac{b_k z^k}{k!}$$
$$= \frac{z}{e^z - 1}$$

We will not prove this since it goes beyond the scope of the book.

## 3. Applications of the Snake Oil Method

*Example.* Consider the sum,

$$\sum_{k\geq 0} \binom{k}{n-k}$$

We will apply the snake oil method to it. First, we take free variable $n$ and let

$$f(n) = \sum_{k\geq 0} \binom{k}{n-k}$$

Now take the generating function

$$F(z) = \sum_{n\geq 0} z^n \sum_{k\geq 0} \binom{k}{n-k}$$

Now we interchange sums,

$$\sum_{k\geq 0} \sum_{n\geq 0} \binom{k}{n-k} z^n = \sum_{k\geq 0} z^k \sum_{n\geq 0} \binom{k}{n-k} z^{n-k}$$

And now we can see that the inner sum can be simplified with the binomial theorem,

$$F(z) = \sum_{k\geq 0} z^k (1+z)^k$$

$$= \sum_{k\geq 0} (z+z^2)^k = \frac{1}{1-z-z^2}$$

From our important generating functions section, we can see that this is the shifted fibonacci generating function, and comparing coefficients we get the identity,

$$\sum_{k\geq 0} \binom{k}{n-k} = f_{n+1}$$

*Example.* We consider the sum,

$$\sum_{k\geq 0} \binom{n+k}{2k} 2^{n-k}$$

We now set $n$ as the free variable and

$$f(n) = \sum_{k\geq 0} \binom{n+k}{2k} 2^{n-k}$$

$$F(z) = \sum_{n\geq 0} z^n \sum_{k\geq 0} \binom{n+k}{2k} 2^{n-k}$$

We now proceed with our method of interchanging sums and solving for coefficients:

$$F = \sum_{k\geq 0} 2^{-k} \sum_{n\geq 0} \binom{n+k}{2k} 2^n z^n$$

$$= \sum_{k} 2^{-k}(2z)^{-k} \sum_{n\geq 0} \binom{n+k}{2k} 2z^{n+k}$$

$$= \sum_{k\geq 0} 2^{-k}(2z)^{-k} \frac{(2z)^{2k}}{(1-2z)^{2k+1}}$$

$$= \frac{1}{1-2z} \sum_{k\geq 0} \left(\frac{z}{(1-2z)^2}\right)^k$$

$$= \frac{1}{1-2z} \frac{1}{1-\frac{z}{(1-2z)^2}}$$

$$= \frac{1-2z}{(1-4z)(1-z)}$$

$$= \frac{2}{3(1-4z)} + \frac{1}{3(1-z)}$$

and computing coefficients we get

$$[z^n]F(z) = \frac{2^{2n+1}+1}{3}$$

and the identity

$$\sum_{k} \binom{n+k}{2k} 2^{n-k} = \frac{2^{2n+1}+1}{3}$$

*Example.* We have the identity,

$$\sum_{k\geq 0} \binom{n}{k} \binom{2n}{n-k} = \binom{3n}{n}$$

We are not going to prove the identity in this example, but show an important snake oil technique. We see that the multiple occurrences of $n$ make using the snake oil method impossible, but to circumvent this we can find a closed form for

$$\sum_{k\geq 0} \binom{n}{k} \binom{m}{r-k}$$

(which is easy to do with the snake oil method), and plug in $r = n, m = 2n$, to get our sum.

*Example.* A numerous amount of combinatorial identities are special cases of hypergeometric series theory identities. The scope of the snake oil method is so large, that it can solve a large amount of sums that hypergeometric series tools can't. One such sum is

$$\sum_{k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix} b_k$$

where $b_k$ are the Bernoulli numbers. First, we take the free variable $n$, and take

$$f(n) = \sum_{k\geq 0} \begin{bmatrix} n \\ k \end{bmatrix} b_k$$

Instead of finding the ordinary generating function of $f(n)$, we try to find the exponential generating function,

$$F(z) = \sum_n \frac{z^n}{n!} \sum_k \begin{bmatrix} n \\ k \end{bmatrix} b_k$$

and proceed with

$$F(z) = \sum_k b_k \sum_k \begin{bmatrix} n \\ k \end{bmatrix} \frac{x^n}{n!}$$

The inner sum we know is

$$\frac{\left(\log \frac{1}{1-z}\right)^k}{k!}$$

We can let $u = \log \frac{1}{1-z}$ and get

$$F(z) = \sum_k b_k \frac{u^k}{k!}$$

which we know is

$$\frac{u}{e^u - 1}$$

$$= \frac{1-z}{z} \log \frac{1}{1-z}$$

Finally, if we take the coefficients, we get

$$n![z^n]F(z) = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} b_k = -\frac{(n-1)!}{n+1}$$