THE CYCLIC SIEVING PHENOMENON AND APPLICATIONS

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ABSTRACT. The cyclic sieving phenomenon is an intriguing idea that has appeared in many combinatorial contexts, such as promotions and the Young Tableaux, pattern-avoiding permutations, and circular Dyck paths. In this paper, we will investigate what the CSP (short for cyclic sieving phenomenon) is and some interesting applications of it.

1. Introduction

Let's say we have a set X, a finite cyclic group C acting on X, and a polynomial $f(q)$ in q with nonnegative integer coefficients.

Definition 1.1 (The Cyclic Sieving Phenomenon)**.** *The triple $(X, C, f(q))$ is said to exhibit the cyclic sieving phenomenon if*

$$\#X^g = f(\omega_{o(g)})$$

where # represents cardinality, X^g is the fixed point set of g, and $\omega_{o(g)}$ is a primitive $o(g)$th root of unity ($o(g)$ denotes the order of g).

At first glance, it may not be immediately clear how plugging a complex number into a polynomial would have any combinatorial meaning. However, it is ubiquitous in many contexts, as we will be seeing in this paper.

As Reiner eloquently states in [wVR], some elements in the set are asymmetric, some have two-fold or three-fold symmetry. Then, plugging in a primitive root of unity of order d would give us a count of how many elements had a d-fold symmetry. Thus, in a sense, the polynomial $f(q)$ is storing information about the cyclic symmetry and orbit structure in this action on this set.

The precursor to the cyclic sieving phenomenon was J. Stembridge's "$q = -1$ phenomenon" [Ste94] which was a special case of the CSP for involutions:

Definition 1.2 (The $q = -1$ phenomenon)**.** *Suppose we have finite set of combinatorial objects X and an associated generating function $F(q)$ (with nonnegative integer coefficients). Furthermore, suppose we have an involution c on X. Then, this setting would exhibit the $q = -1$ phenomenon if $F(1)$ would be equal to the number of elements in X and $F(-1)$ would be equal to the number of fixed points in X under c. In other words, if*

$$F(1) = \#X \text{ and } F(-1) = \#X^c.$$

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2. Investigating a CSP Triple

Let's look at a triple that exhibits the CSP. For some positive integer n, let $X = \left(\!\!\binom{[n]}{k}\!\!\right)$, the set of multisets of $[n]$. Now, consider the group $G = \langle(1, 2, \dots, n)\rangle$. We let G act on elements of X by the following definition:

Definition 2.1. *Let $g \in G$ and $M \in X$ such that $M = \{a_1, a_2, a_3, \dots, a_k\}$ where $a_1 \leq a_2 \leq \cdots \leq a_k$. Then,*

$$g(M) = \{g(a_1), g(a_2), \dots g(a_k)\}.$$

For example, let's look at the case where $n = 3$ and $k = 2$. Then,

$$X = \{\{1, 1\}, \{1, 2\}, \{1, 3\}, \{2, 2\}, \{2, 3\}, \{3, 3\}\}.$$

Also, we have

$$G = \{e, (1, 2, 3), (1, 3, 2)\}$$

where e is the identity transformation. Let's say $g = (1, 2, 3)$, i.e. $g(1) = 2$, $g(2) = 3$, and $g(3) = 1$. Then, we have

$$\begin{aligned} g(X) = (1, 2, 3)(X) = \{&g(\{1, 1\}), \quad g(\{1, 2\}), \quad g(\{1, 3\}), \\ &g(\{2, 2\}), \quad g(\{2, 3\}), \quad g(\{3, 3\}) \\ = \{&\{2, 2\}, \quad \{2, 3\}, \quad \{2, 1\}, \\ &\{3, 3\}, \quad \{3, 1\}, \quad \{1, 1\}\}. \end{aligned}$$

Now, the last ingredient we need is to define a function f. For this, we are going to use *q-analogs*:

Definition 2.2. *The q-analog of a positive integer m is*

$$[m]_q = 1 + q + q^2 + q^3 + \cdots + q^{m-1}.$$

Furthermore, let the q-analog of $m!$ be

$$[m]!_q = [m]_q[m-1]_q \cdots [2]_q[1]_q.$$

Finally, define the q-analog of the binomial coefficient $\binom{m}{j}$ to be

$$\begin{bmatrix} m \\ k \end{bmatrix}_q = \frac{[m]!_q}{[k]!_q[m-k]!_q}.$$

Remark 2.1. The formula for the q-analog of $[m]_q$ is motivated by the fact that

$$\lim_{q \to 1} \frac{1 - q^m}{1 - q} = m$$

for any positive integer m [ALP19].

We have the following theorem by Reiner, Stanton, and White [RSW14]:

Theorem 1 (Reiner-Stanton-White)**.** *The triple*

$$\left(\left(\!\!\binom{[n]}{k}\!\!\right), \langle(1, 2, \dots, n)\rangle, \begin{bmatrix} n + k - 1 \\ k \end{bmatrix}_q \right)$$

exhibits the CSP.

To prove this theorem, we will use the proof displayed by Bruce Sagan in [Sag10]:

Proof. First off, we will start out by proving the following lemma, which will later help us understand what is going on on the right-hand side of the equation:

Lemma 2.1. *For positive integers m and n such that $m \equiv n \pmod d$,*

$$\lim_{q\to\omega}\frac{[m]_q}{[n]_q} = \left\{\begin{array}{ll} \frac{m}{n}, & \text{for } m\equiv n\equiv 0 \pmod d \\ 1, & \text{otherwise}\end{array}\right\}$$

where ω is a root of unity with order d.

To prove this lemma, we start out by letting $m \equiv n \equiv r \pmod d$ where $0 \le r < d$. Then, we have

$$\begin{aligned}[m]_\omega &= 1+\omega+\omega^2+\cdots+\omega^{m-1}\\ &= (1+\omega+\cdots+\omega^{d-1})+\cdots+(1+\omega+\cdots+\omega^{d-1})+(1+\omega+\cdots+\omega^{r-1})\\ &= 1+\omega+\cdots+\omega^{r-1}\end{aligned}$$

since $1+\omega+\cdots+\omega^{d-1} = 0$. Therefore, we also have $[n]_\omega = [m]_\omega = 1+\omega+\cdots+\omega^{r-1}$.

If $n \not\equiv 0 \pmod d$, then $[n]_\omega \neq 0$. Thus, $\frac{[m]_\omega}{[n]_\omega} = 1$.

Now, if $n \equiv 0 \pmod d$, then $[n]_\omega = 0$. In this case, let $m = jd$ and $n = kd$. Then, we have

$$\begin{aligned}\frac{[m]_q}{[n]_q} &= \frac{1+q+q^2+\cdots+q^{jd-1}}{1+q+q^2+\cdots+q^{kd-1}}\\ &= \frac{(1+q+\cdots+q^{d-1})+\cdots+(1+q+\cdots+q^{d-1})+(1+q+\cdots+q^{r-1})}{(1+q+\cdots+q^{d-1})+\cdots+(1+q+\cdots+q^{d-1})+(1+q+\cdots+q^{r-1})}\\ &= \frac{(1+q+\cdots+q^{d-1})(1+q^d+q^{2d}+\cdots+q^{(j-1)d})}{(1+q+\cdots+q^{d-1})(1+q^d+q^{2d}+\cdots+q^{(k-1)d})}\\ &= \frac{1+q^d+q^{2d}+\cdots+q^{(j-1)d}}{1+q^d+q^{2d}+\cdots+q^{(k-1)d}}.\end{aligned}$$

Taking the limit as q approaches ω and using the fact that $\omega^d = 1$, we have

$$\begin{aligned}\lim_{q\to\omega}\frac{[m]_q}{[n]_q} &= \lim_{q\to\omega}\frac{1+q+q^2+\cdots+q^{jd-1}}{1+q+q^2+\cdots+q^{kd-1}}\\ &= \frac{1+\omega^d+\omega^{2d}+\cdots+\omega^{(j-1)d}}{1+\omega^d+\omega^{2d}+\cdots+\omega^{(k-1)d}}\\ &= \frac{j}{k} = \frac{m}{n}.\end{aligned}$$

Now, we propose the following equality regarding the right-hand side of the CSP condition:

Proposition 2.1. *Let $n, d,$ and k be positive integers such that $d|n$. Then,*

$$\lim_{q\to\omega}\begin{bmatrix} n+k-1 \\ k\end{bmatrix}_q = \left\{\begin{array}{ll} \binom{n/d+k/d-1}{k/d}, & \text{for } d|k \\ 0, & \text{otherwise}\end{array}\right\}$$

The proof for this proposition revolves around noticing the following: since $d|n$, the number of factors equal to zero in the numerator of the expression when $q = \omega$, which is $[n]_\omega[n+1]_\omega[n+2]_\omega\cdots[n+k-1]_\omega$, is always greater than or equal to the number of zeros in the denominator, which is $[k]_\omega[k-1]_\omega\cdots[2]_\omega[1]_\omega$. Equality is achieved if and only if $d|k$. This proves the else case.

If $d|k$, then

$$\begin{aligned}\lim_{q\to\omega}\begin{bmatrix}n+k-1\\ k\end{bmatrix}_\omega &= \lim_{q\to\omega}\left(\frac{[n]_q}{[k]_q}\frac{[n+1]_q}{[1]_q}\frac{[n+2]_q}{[2]_q}\cdots\frac{[n+k-1]_q}{[k-1]_q}\right)\\ &= \frac{n}{k}\cdot 1\cdots 1\cdot\frac{n+d}{d}\cdot 1\cdots 1\cdot\frac{n+2d}{2d}\cdots\frac{n+k-d}{k-d}\cdot 1\cdots\\ &= \frac{n/d}{k/d}\frac{n/d+1}{1}\frac{n/d+2}{2}\cdots\frac{(n+k)/d-1}{k/d-1}\\ &= \binom{(n+k)/d-1}{k/d}\end{aligned}$$

where the second equality comes from application of Lemma 3.1.

Now, we understand what happens on the right-hand side of the equation when we plug in our root of unity ω. Let's take a look at what happens at the left-hand side:

Proposition 2.2. *Let* $M=\left(\!\binom{[n]}{k}\!\right)$, *and let* $o(g)=d$. *Then,*

$$\#M^g=\left\{\begin{array}{ll}\binom{n/d+k/d-1}{k/d}, & \text{if } d|k\\ 0, & \text{otherwise}\end{array}\right\}$$

Let's consider a simple case for our proof: $g=(1,2,\ldots,n)^{n/d}$. Then, the proof for other possible g only requires slight modification from the current one, thus we leave that as an exercise to the reader.

Considering the cycle decomposition of g, we have

$$g=(1,1+\frac{n}{d},1+2\frac{n}{d},\cdots)(2,2+\frac{n}{d},2+2\frac{n}{d},\cdots)\cdots(n,n+\frac{n}{d},n+2\frac{n}{d},\cdots).$$

Let g_1 denote the set of elements in first cycle, g_2 denote the set of elements in the second cycle, g_3 the set of elements in the third cycle, etc. Now, notice that for any $X\in M$, X is a fixed point if and only if we can express it as

$$X=g_{i_1}\uplus g_{i_2}\uplus\cdots$$

for some positive integers $i_1,i_2,\ldots$, where $\uplus$ denotes the multiset sum of two sets. In other words, X is the mulstiset sum of some (and possibly all) of these cycles.

Consider if $X=g_1\uplus g_2\uplus\cdots$. Then, gX would simply be g acting on a bunch of cycles, which would shift over each element in every cycle over. In other words, we are just rearranging the elements within the same cycle, but the elements themselves are staying the same. Thus, $gX=X$.

Conversely, let's say we started out with $gX=X$. Consider a specific element, for example $j\in X$. Then, we know that $j+n/d\in X$. However, if this is the case, then $j+2n/d\in X$ as well. Continuing this process, we see that all elements in that cycle are also in X.

Now, remember that the order of g is d, therefore each of the cycles must have d elements. Consequently, if d does not divide k, then the multiset cannot be written in the form above, which implies that $\#M^g=0$ since it has no fixed points subsets.

However, if we do have $d|k$, then we need exactly $\frac{k}{d}$ cycles to cover all of the elements. Since we have $\frac{n}{d}$ total cycles to choose from (with repetition), we have

$$\#M^g=\left(\!\!\binom{n/d}{k/d}\!\!\right),$$

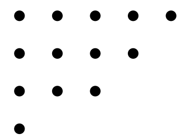

FIGURE 1. Ferrers diagram for the partition $(5,4,3,1)$.

which is equal to

$$\binom{n/d + k/d - 1}{k/d}.$$

Therefore, we have shown that both the left-hand side and right-hand side of the equation are counting the same quantity, completing the proof.

□

3. Applications

In the applications section, we will be going over how CSP applies to Young Tableaux, Pattern-Avoiding Permutations, and Circular Dyck Paths.

3.1. **Cyclic Sieving in the Young Tableaux.**

In his paper titled "Cyclic sieving, promotion, and representation theory," B. Rhoades [Rho10] provides another cyclic sieving triple about regular Young Tableaux under the action of promotion. Before we continue, let us review some definitions.

A *partition* λ of n is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of k non-increasing positive integers such that $\sum_i \lambda_i = n$. We denote this as $\lambda \vdash n$, and λ_i are the *parts* of λ. For example, $(5,4,3,1)$ is a partition of 13 with parts $5,4,3$, and 1.

Given a partition λ of n, its *Ferrers diagram* is a way to represent the partition in which we place λ_i dots in row λ_i. For example, the partition $(5,4,3,1)$ has the corresponding Ferrers diagram shown in Figure 1.

We say λ is a *regular partition* if it can be expressed in the form of $\lambda = (n, n, \ldots, n) = (n^m)$, i.e. a partition of a number into m parts, each of size n. Note that regular partitions have rectangular-shaped Ferrers diagrams.

Now, define the *standard Young tableaux* of shape λ to be a function $T : \lambda \to [n]$ such that $T_{i,j} < T_{i,j+1}$ and $T_{i,j} < T_{i+1,j}$ where $T_{x,y}$ is used to represent the element in the x^{th} row and y^{th} column. We let $\mathrm{SYT}(\lambda)$ denote all of the standard Young tableaux of shape λ, and define

$$\mathrm{SYT}_n = \bigcup_{\lambda \vdash n} \mathrm{SYT}(\lambda).$$

For instance, the possible standard Young tableaux for the partition $\lambda = (3,2)$ of 5 are

$$\mathrm{SYT}((3,2)) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline 4 & 5 \\ \cline{1-2}\end{array}\ , \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 \\ \cline{1-2}\end{array}\ , \begin{array}{|c|c|c|}\hline 1 & 2 & 5 \\ \hline 3 & 4 \\ \cline{1-2}\end{array}\ , \begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 \\ \cline{1-2}\end{array}\ , \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 \\ \cline{1-2}\end{array} \right\}$$

So $\#\mathrm{SYT}((3,2)) = 5$. If λ is a regular partition, then its standard Young tableaux are also *regular Young tableaux*.

Furthermore, we define

$$f^\lambda = \#\textsc{Syt}(\lambda).$$

Recall that a cyclic sieving triple must include a set, a polynomial, and a group acting on that set. So far, we have found the set that Rhoades uses in his theorem: $\textsc{Syt}(\lambda)$. To find the desired polynomial, we look to the concept of *hooklength* as is introduced by J. S. Frame, G. de B. Robinson, and R. M. Thrall [FRT54].

We define the *hook* of (i, j) in the Young tableaux to be

$$H_{i,j} = \{(i, j') \in \lambda : j' \geq j\} \cup \{(i', j) \in \lambda : i' \geq i\}$$

and its corresponding *hooklength* to be

$$h_{i,j} = \#H_{i,j}.$$

The fascinating result discovered by Frame, Robinson, and Thrall that comes out of this is called the Hooklength Formula:

Theorem 2 (Frame-Robinson-Thrall)**.** *For a partition $\lambda \vdash n$, we have*

$$f^\lambda = \frac{n!}{\prod\limits_{(i,j)\in\lambda} h_{i,j}}.$$

Thus, the polynomial that appears in Rhoades' cyclic sieving triple is the q-analog of the Hooklength Formula:

$$f^\lambda(q) = \frac{[n]!_q}{\prod\limits_{(i,j)\in\lambda} [h_{i,j}]_q}.$$

The last thing we need now is the group action on $\textsc{Syt}(\lambda)$: the perfect candidate for this is using Schützenberger's promotion operator [Sch72]. First, define $(i, j) \in \lambda$ to be a *corner* if $(i, j+1) \notin \lambda$ and $(i+1, j) \notin \lambda$. Now, for a $T \in \textsc{Syt}(\lambda)$, define the *promotion* ∂T of T via the following algorithm:

(1) Replace $T_{1,1}$ with a dot.
(2) If the dot is at (i, j), then swap it with the value at $T_{i+1,j}$ or $T_{j+1,i}$, whichever one is smaller. If only one of them is in T, then swap it with that one. Repeat this step until the dot is at a corner.
(3) Subtract 1 from every element in the array, and replace the dot in the corner with the value n. This gives us ∂T.

For example, to get the promotion of $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array}$, we would follow this sequence of steps:

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline \cdot&2&4\\\hline 3&5\\\cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 2&\cdot&4\\\hline 3&5\\\cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 2&4&\cdot\\\hline 3&5\\\cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&\cdot\\\hline 2&4\\\cline{1-2}\end{array} \to \begin{array}{|c|c|c|}\hline 1&3&5\\\hline 2&4\\\cline{1-2}\end{array}$$

Note that the second step simply shifts the elements of the tableaux, so it maintains the property that the elements are in increasing order from left-to-right and from top-to-bottom. Therefore, $\partial T \in \textsc{Syt}(\lambda)$. Additionally, it is not difficult to see that this algorithm is reversible at every step, therefore ∂ is a bijection on $\textsc{Syt}(\lambda)$ (we encourage the reader to pick a λ, then pick a few $T \in \textsc{Syt}(\lambda)$ and show that ∂T is a bijection). Therefore, the promotion operator generates a group $\langle \partial \rangle$ acting on Young tableaux of a given shape.

Though mapping out the action in general seems challenging, it is easier to describe the action for some special shapes. The following theorem is by M. D. Haiman [Hai92], which shows that $\langle\partial\rangle$ is cyclic when acting on regular Young tableaux:

Theorem 3 (Haiman)**.** *If $\lambda = (n, n, \ldots, n) = (n^m)$ (a partition of mn into m equal parts of size n), then*

$$\partial^{mn} T = T$$

for all $T \in \mathrm{SYT}(\lambda)$.

In fact, Sagan [Sag10] provides the cycle decomposition for ∂ via its action on $\lambda = \left(3^2\right)$.

However, the main result we get from Rhoades' paper is the following:

Theorem 4 (Rhoades)**.** *If $\lambda = (n^m)$, then the triple*

$$\left(\mathrm{SYT}(\lambda), \langle\partial\rangle, f^{\lambda}(q)\right)$$

exhibits the cyclic sieving phenomenon.

This result inspired many more papers which furthered this idea. For example, Rhoades later wrote a paper with TK Petersen and P Pylyavskyy [PPR09] which also incorporated the concept of webs.

3.2. **The $q = -1$ Phenomenon for 132 Pattern-Avoiding Permutations.** We continue our discussion on examples of the CSP's ubiquity in combinatorics by taking a look at a more specific case of the CSP. As discussed earlier, Stembridge's $q = -1$ phenomenon [Ste94] was the precursor to the generalized Cyclic Sieving Phenomenon investigated by Reiner, Stanton, and White. We will take a look at how this specialized version appears in counting pattern-avoiding permutations as is presented by X. Chen [Che11].

Let us begin with some basic definitions. Firstly, a *permutation* of $[n]$ is a bijection from $[n]$ to itself. We let S_n represent the set of all permutations of $[n]$. A permutation $\pi \in S_n$ that satisfies the property $\pi(\pi(i)) = i$ for all $i \in [n]$ is called an *involution*.

Furthermore, given another permutation σ such that $\sigma \in S_k$ for some $k \leq n$, we say that a subsequence $\pi(i_1), \pi(i_2), \ldots, \pi(i_k)$ of π is σ *type* if $i_l < i_r$ implies $\pi(i_l) < \pi(i_r)$ if and only if $\sigma(i_l) < \sigma(i_r)$. We say that σ is a *subpermutation* of π if there is some subsequence of π that is σ type. For example, 3412 is a subpermutation 156324, since the subsequence 5634 is 3412 type. We say π avoids σ if σ is not a subpermutation of π. Given a set R of permutations, we denote the set of all permutations of length n that avoid every permutation in R by $S_n(R)$, and use $S(R) = \bigcup_{n \in \mathbb{Z}_{\geq 1}} S_n(R)$.

Now, we have one last definition: let's say we take a permutation $\pi \in S_n$ for some positive integer n, and divide it up into blocks of increasing runs; for example we would split the permutation $\pi = 13782546$ into $\pi = 1378|25|46$. Now, for all $1 \leq i \leq n$, let $r(i)$ be defined as the number of blocks to the right of position i that contain numbers both greater then $\pi(i)$ and smaller than $\pi(i)$. Then, define the statistic $\mathrm{rsg}(\pi)$ by

$$\mathrm{rsg}(\pi) = \sum_{i=1}^{n} r(i).$$

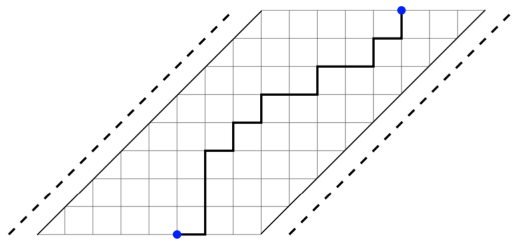

FIGURE 2. An example of a Circular Dyck Path. Image from [ALP19].

Returning to our example of $\pi = 1378|25|46$, we have

$$\text{rsg}(\pi) = 0 + 1 + 0 + 0 + 0 + 1 + 0 + 0 = 2.$$

Alternatively but equivalently, we could also define $\text{rsg}(\pi)$ as the number of $2-13$ type permutations (the dash represents possible numbers in between the 2 and 1, however the 1 and 3 must be within the same block). Thus, we have two $2-13$ type permutations above, namely $3-25$ and $5-46$.

Now, we can define the polynomial that will later come in play in the theorem:

$$F_n(q) = \sum_{\pi \in S_n(132)} q^{\text{rsg}(\pi)}.$$

For instance, it can be shown that $F_3(q) = 4 + q$ and $F_4(q) = 8 + 4q + 2q^2$. We leave it as an exercise to the reader to show that this is in fact true.

The main result of [Che11] is the following:

Theorem 5. *The number of involutions in* $S_n(132)$ *is* $F_n(-1)$.

Consider as an example $n = 3$. Then, the involutions in $S_3(132)$ are $123, 213$, and 321. Also, notice that $F_3(-1) = 4 + (-1) = 3$, so $n = 3$ does satisfy the theorem.

For completeness, one could also show that $F_n(1) = \#S_n(132)$, which is the other statement in Setmbridge's theorem [Ste94]. However, this is fairly trivial, since the sum of the coefficients of $F_n(q)$ is simply the total number of 132-avoiding permutations over all possible rsg statistic values, which would give us all 132-avoiding permutations.

3.3. **The CSP for Circular Dyck Paths.**

Now, we return back to the general scope of the CSP and see how it applies to Circular Dyck Paths (CDPs). The theorem was presented by Alexandersson, Linusson, and Potka [ALP19] which we will be investigating here.

Firstly, a *Circular Dyck Path* can be thought of as a Dyck path that wraps around and ends at the same point from which it started. An example of a Circular Dyck Path is in Figure 2. Notice how the CDP is in a paralellogram shape, since by definition of a Dyck path, the CDP cannot cross the diagonal border.

Every CDP (and every regular Dyck path for that matter) can be described by an *area sequence*, which specifies the number of squares between the path and the right edge of the parallelogram. We can formally define an area sequence in the following manner:

Definition 3.1. *A Circular Dyck Path of height n and width m can be specified by an* area sequence $(a_1, a_2, \ldots, a_n)$ *that satisfy*

- $a_i \in \mathbb{Z}_{\geq 0}$,
- $0 \leq a_i \leq m-1$, *and*
- $a_{i+1} \leq a_i + 1$ *(we consider $a_{n+1} = a_1$)*

for all $1 \leq i \leq n$. The set of all such sequences is denoted as CDP(n, m).

For example, the CDP in Figure 2 has an area sequence of $(2, 3, 4, 4, 4, 3, 2, 2)$. Another way to think about CDPs is from a starting point $(x_0, 0)$ such that $1 \leq x_0 \leq m$ then moving in a sequence of eastward or upward steps specified by a binary string of length $n + m$, where a 0 represents an eastward move and a 1 represents an upward move.

The natural group action on the set CDP(n, m) is simply cyclically shifting the area sequence by 1. We will denote such a shift as α. α will come in play later when we are defining the cyclic group acting on CDP(n, m).

The last step is to find the q-polynomial which would work best for our CSP triple. As it turns out, after much algebra and computation, Alexandersson-Linusson-Potka found that the desired q-enumeration is

$$|\mathrm{CDP}(n,m)|_q = \sum_{s\in\mathbb{Z}}\sum_{j=1}^{m} q^{s^2\delta+s(j+1)}\left(\begin{bmatrix}2n-1\\ n-1-\delta s\end{bmatrix}_q - \begin{bmatrix}2n-1\\ n+j+\delta s\end{bmatrix}_q\right)$$

where the substitution $\delta = m + 2$ is being made. In particular, if $m \geq n$, then

$$|\mathrm{CDP}(n,m)|_q = m\begin{bmatrix}2n-1\\ n-1\end{bmatrix}_q - \sum_{j=1}^{m} q^j \begin{bmatrix}2n-1\\ n+j\end{bmatrix}_q - \sum_{j=1}^{m}\begin{bmatrix}2n-1\\ n+j-(m+2)\end{bmatrix}.$$

Remark 3.1. Some combinatorial identities can be proved using q-analogs. For example, Alexandersson-Linusson-Potka proved the identity

$$\begin{aligned}\#\mathrm{CDP}(n,m) &= \sum_{t\in\mathbb{Z}}\left[\binom{2n-1}{n+(m+2)t} - \binom{2n-1}{n+t}\right]\\ &= \sum_{t\in\mathbb{Z}}\binom{2n-1}{n+(m+2)t} - 2^{2n-1}\end{aligned}$$

simply by letting $q = 1$ in the q-enumeration above and then adding some slight modifications.

We have all of the ingredients we need for our CSP triple:

Theorem 6 (Alexandersson-Linusson-Potka)**.** *The triple*

$$(CDP(n,m), \langle\alpha\rangle, |CDP(n,m)|_q)$$

exhibits the cyclic sieving phenomenon.

Further variants of the CSP are explored in [ALP19], such as subset cyclic sieving and the introduction of Lyndon-like cyclic sieving.

4. Final Remarks

The cyclic sieving phenomenon is a beautiful result that appears in many seemingly unrelated aspects of combinatorics, all the way from standard Young tableaux to pattern-avoiding permutations to Circular Dyck Paths. There are many more examples of fields where the CSP appears, such as non-crossing trees, edges, and graphs [Poz11]; words [RSW14]; and more. It is yet to see in what other intriguing locations this phenomenon is discovered. In the mean time, we continue to ask ourselves new questions and variants on the cyclic sieving phenomenon, continuously exploring its natural ubiquity.

5. Acknowledgements

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