HYPERGEOMETRIC FUNCTIONS

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1. Introduction

If you ever try to find a general term for a particular generating function, say, for the number of bracketings of a sequence of identical letters (see Week 3 Problem 3),

$$s_n = \frac{1}{n}\sum_{k=1}^{n-1} 2^{k-1}\binom{n}{k}\binom{n-2}{k-1},$$

you will not always get an explicit formula for the nth coefficient. Now your natural instinct might be to plug such a result into WolframAlpha to see if they have a nicer answer, and in this case we get

$$s_n = {}_2F_1(1-n, 2-n; 2; 2).$$

In this paper, we will be exploring what the above result means and how to make sense of it. To start, we present some basic definitions.

Definition 1.1. A *hypergeometric series* is any power series whose ratios between consecutive coefficients $\frac{\alpha_n+1}{\alpha_n}$ satisfy a rational function in terms of n. All hypergeometric series are hence of the form

$$\sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{\prod_{k=1}^{p}(j+a_k-1)}{\prod_{k=1}^{q}(j+b_k-1)}\right)\frac{z^n}{n!}.$$

For sake of clarity, we can shorten the above expression into

$${}_pF_q(a_1, a_2, \ldots, a_p; b_1, b_2, \ldots, b_q; z).$$

Definition 1.2. The *hypergeometric function* is one of the most common forms of such hypergeometric series:

$${}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j+a-1)(j+b-1)}{(j+c-1)}\right)\frac{z^n}{n!}.$$

Notice that a, b, and c need not be integers, or even constants, as seen in our example with the bracketings. We will be working closely with the hypergeometric function to discover many of its properties and also learn how to write many of the most basic functions we work with in terms of such a general formula.

Interestingly enough, this hypergeometric function is the solution to the differential equation $z(z-1)f'' + ((a+b+1)z - c)f' + abf = 0$.

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2. Special Properties

We begin our exploration of these functions by looking at some basic properties.

Theorem 2.1. *In ${}_2F_1(a,b;c;z)$, if either a or b is a nonpositive integer, the function converges.*

Proof. Without loss of generality assume that a is a nonpositive integer. Then starting from the index $n = 1 - a$, the numerator will be equal to 0 since $(n + a - 1) = 0$. As a result, none of the terms that follow $n = 1 - a$ will contribute anything to the sum either, and we end up with a finite series of terms:

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{1-a} \left(\prod_{j=1}^{n} \frac{(j+a-1)(j+b-1)}{(j+c-1)} \right) \frac{z^n}{n!}. $$

□

It is also worthwhile to compute the derivative of the hypergeometric function, as it gives us a surprisingly simple result:

$$ \begin{aligned} \frac{d}{dz} {}_2F_1(a,b;c;z) &= \frac{d}{dz} \sum_{n=0}^{\infty} \left(\prod_{j=1}^{n} \frac{(j+a-1)(j+b-1)}{(j+c-1)} \right) \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\prod_{j=1}^{n} \frac{(j+a-1)(j+b-1)}{(j+c-1)} \right) \frac{z^{n-1}}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \left(\prod_{j=1}^{n+1} \frac{(j+a-1)(j+b-1)}{(j+c-1)} \right) \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{ab}{c} \prod_{j=2}^{n+1} \frac{(j+a-1)(j+b-1)}{(j+c-1)} \right) \frac{z^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\frac{ab}{c} \prod_{j=1}^{n} \frac{(j+(a+1)-1)(j+(b+1)-1)}{(j+(c+1)-1)} \right) \frac{z^n}{n!} \\ &= \frac{ab}{c} {}_2F_1\left(a+1, b+1; c+1; z\right). \end{aligned} $$

3. Elementary Functions using ${}_2F_1$

We start off with the following identity:

Theorem 3.1.

$$ \log(1+z) = z {}_2F_1(1,1;2,-z). $$

Proof. Computing the right hand side gives us

$$
\begin{aligned}
z_2F_1(1,1;2,-z) &= \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j+1-1)(j+1-1)}{(j+2-1)}\right)\frac{(-z)^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{j^2}{(j+1)}\right)\frac{(-z)^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(n!)^2}{(n+1)!}\cdot\frac{(-z)^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(-z)^n}{n}\\
&= \log(1+z),
\end{aligned}
$$

as desired. □

Theorem 3.2.

$$(1-z)^a = {}_2F_1(-a,x;x;z).$$

Proof. Once again, we just compute the right hand side.

$$
\begin{aligned}
z_2F_1(-a,x;x,z) &= \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j-a-1)(j+x-1)}{(j+x-1)}\right)\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}(j-a-1)\right)\frac{z^n}{n!}\\
&= \sum_{n=0}^{\infty}\frac{(n-a-1)(n-a-2)\cdots(-a)}{(n)(n-1)\cdots(1)}z^n\\
&= \sum_{n=0}^{\infty}\frac{(-1)^n(a)(a-1)\cdots(a-n+1)}{(n)(n-1)\cdots(1)}z^n\\
&= \sum_{n=0}^{\infty}\binom{a}{n}(-z)^n\\
&= (1-z)^a,
\end{aligned}
$$

as desired. □

Theorem 3.3.

$$\arctan(z) = z_2F_1(1/2,1;3/2;-z^2).$$

Proof. For the last time, we proceed as follows:

$$\begin{aligned} z\,{}_2F_1(1/2,1;3/2;-z^2) &= z\sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{\left(j+\frac{1}{2}-1\right)(j+1-1)}{\left(j+\frac{3}{2}-1\right)}\right)\frac{(-z^2)^n}{n!} \\ &= z\sum_{n=0}^{\infty}\left(\frac{n!}{2n+1}\right)\frac{(-z^2)^n}{n!} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(z)^{2n+1}}{2n+1} \\ &= \arctan(z), \end{aligned}$$

where we know that

$$\prod_{j=1}^{n}\frac{\left(j+\frac{1}{2}-1\right)(j+1-1)}{\left(j+\frac{3}{2}-1\right)} = \frac{(1/2\cdot 3/2\cdot 5/2\cdots(2n-1)/2)n!}{(3/2\cdot 5/2\cdots(2n-1)/2)\cdot(2n+1)/2} = \frac{n!}{2n+1}.$$

□

4. Chu-Vandermonde Identity

We now endeavor to prove some identities regarding the hypergeometric function in order to ultimately derive the Chu-Vandermonde Identity. Starting with arbitrary z, we have

$$(1-zt)^{-a} = \sum_{n=0}^{\infty}\frac{(-1)^n(a)(a+1)\cdots(a+n-1)}{n!}(-z)^n t^n = \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j+a-1)}{j}\right)z^n t^n.$$

This allows us to write the following:

$$\begin{aligned} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt &= \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j+a-1)}{j}\right)z^n t^n\int_0^1 t^{n+b-1}(1-t)^{c-b-1}dt \\ &= \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j+a-1)}{j}\right)z^n t^n B(n+b,c-b), \end{aligned}$$

where $B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt$ is the Beta function. There exists an identity that $B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, where Γ is the regular Gamma function. Thus, we can rewrite the above integral as

$$\frac{\Gamma(n+b)\Gamma(c-b)}{\Gamma(n+c)}.$$

Thus, we obtain

$$\begin{aligned}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}(1-zt)^{-a}\int_0^1 t^{b-1}(1-t)^{c-b-1}dt\\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j+a-1)}{j}\right)z^n\frac{\Gamma(n+b)\Gamma(c-b)}{\Gamma(n+c)}\\ &= \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j+a-1)}{j}\right)z^n\frac{\Gamma(c)\Gamma(n+b)}{\Gamma(b)\Gamma(n+c)}\\ &= \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j+a-1)(j+b-1)}{j(j+c-1)}\right)z^n\\ &= \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j+a-1)(j+b-1)}{(j+c-1)}\right)\frac{z^n}{n!}\\ &= {}_2F_1(a,b;c;z).\end{aligned}$$

Importantly, this identity gives us a cleaner expression for a special case as well:

$$\begin{aligned}{}_2F_1(a,b;c;1) &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}dt\\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-a-1}dt\\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}B(b,c-a-b)\\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\cdot\frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)}\\ &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}.\end{aligned}$$

Letting $a=-m$ we get

$$\begin{aligned}{}_2F_1(-m,b;c;1) &= \frac{\Gamma(c)\Gamma(c+m-b)}{\Gamma(c+m)\Gamma(c-b)}\\ &= \frac{\Gamma(c)}{\Gamma(c+m)}\cdot\frac{\Gamma(c+m-b)}{\Gamma(c-b)}\\ &= \frac{1}{(c)(c+1)\cdots(c+m-1)}\cdot\frac{(c-b)(c-b+1)\cdots(c-b+m-1)}{1}\\ &= \frac{(c-b)(c-b+1)\cdots(c-b+m-1)}{(c)(c+1)\cdots(c+m-1)},\end{aligned}$$

allow us to eliminate all the Γ's and integrals. Our final step is now to prove the following identity and then use it to our advantage.

Theorem 4.1.

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k}.$$

Proof. We simply compute the right side, as we have always done:

$$\begin{aligned}(-1)^k \binom{k-n-1}{k} &= (-1)^k \frac{(k-n-1)(k-n-2)\cdots(-n)}{k!} \\ &= (-1)^k(-1)^k \frac{(n-k+1)(n-k+2)\cdots(n)}{k!} \\ &= \binom{n}{k},\end{aligned}$$

as expected. □

We can now expand the hypergeometric function to get

$$\frac{(c-b)(c-b+1)\cdots(c-b+m-1)}{(c)(c+1)\cdots(c+m-1)} = {}_2F_1(-m,b;c;1) = \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j-m-1)(j+b-1)}{(j+c-1)}\right)\frac{1}{n!}$$

$$\frac{(c-b)(c-b+1)\cdots(c-b+m-1)}{(c)(c+1)\cdots(c+m-1)} = \sum_{n=0}^{\infty}\left(\prod_{j=1}^{n}\frac{(j-m-1)(j+b-1)}{(j+c-1)j}\right).$$

Now we can use the identity

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k}$$

multiple times to reach the general form of the well-known Vandermonde Identity in combinatorics.

Theorem 4.2 (Chu-Vandermonde Identity). *For complex s, t and nonnegative integers n, we have*

$$\binom{s+t}{n} = \sum_{k=0}^{n}\binom{s}{k}\binom{t}{n-k}.$$

5. Conclusion

We can use hypergeometric series, in particular, ${}_2F_1$, to express a variety of generating functions that otherwise cannot be stated very neatly. We can also see that the hypergeometric function can easily be applied to much simpler series, such as binomial coefficients orthe arctangent function. Finally we saw how a result of Gauss could be used, with the help of the hypergeometric function, to obtain one of the common identities in competition math—Vandermonde. Hypergeometric series have many other applications, such as in mechanics (for example, harmonic oscillation) and atoms.

6. References

Hypergeometric Functions by Roelok Koekoek
https://homepage.tudelft.nl/11r49/documents/wi4006/hyper.pdf