Euler Circle Paper

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1 Summary

The snake oil method is a method used to solve large combinatorial sums. This method works by writing out the generating function of a sum, then reversing the summation to simplify its generating function.

For example, let us consider the sum $\sum_k \binom{k}{n-k}$. Let this sum be f_n. Then, we can write its generating function $F(z)$ as:

$$F(z) = \sum_n f_n z^n = \sum_n \sum_k \binom{k}{n-k} z^n$$

To proceed from here, we must reverse the summation:

$$F(z) = \sum_k \sum_n \binom{k}{n-k} z^n = \sum_k z^k \sum_n \binom{k}{n-k} z^{n-k}$$

Now, in order for the snake oil method to work, the inner summation must simplify to something nice. Otherwise, there is no way to proceed from here. Fortunately, in this case, $\sum_n \binom{k}{n-k} z^{n-k} = (1+z)^k$. With this simplification, we have:

$$F(z) = \sum_k z^k (1+z)^k = \sum_k (z(1+z))^k = \frac{1}{1-z-z^2}$$

This is the generating function of the Fibonacci numbers, thus, we have that $\sum_k \binom{k}{n-k}$ is equal to the nth Fibonacci number.

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Now, let us consider the general case, where $f_n = \sum_k g_{n,k}$. Then,

$$F(z) = \sum_n f_n z^n = \sum_n \sum_k g_{n,k} z^n$$

Reversing the summation,

$$F(z) = \sum_k \sum_n g_{n,k} z^n = \sum_k z^k \sum_n g_{n,k} z^{n-k}$$

Now here we can see what property $g_{n,k}$ must satisfy. We must be able to write the generating function of $\sum_n g_{n,k} z^{n-k}$. Thus, let:

$$G_k(z) = \sum_n g_{n,k} z^{n-k}$$

Then,

$$F(z) = \sum_k z^k G_k(z)$$

And again, we need to get "lucky" and have the generating function $\sum_k z^k G_k(z)$ have a nice form. If it does, then, we can extract the coefficient to get

$$f_n = [z^n] \sum_k z^k G_k(z)$$

So, to summarize, we need to following quantities to be nice in order for the snake oil method to work:

$$\sum_n g_{n,k} z^{n-k}$$

$$\sum_k z^k G_k(z)$$

And, of course,

$$[z^n] \sum_k z^k G_k(z)$$

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We can also write:

$$G_k(z) = \sum_n g_{n+k,k} z^n$$

This form is more useful in many cases. Let us now investigate another example:

$$f_n = \sum_k k^(n-k)$$

Then,

$$G_k(z) = \sum_n g_{n+k,k} z^n = \sum_n k^n z^n = \frac{1}{1-kz}$$

Further,

$$F(z) = \sum_k z^k G_k(z) = \sum_k \frac{z^k}{1-kz}$$

Even though f_n has no easily-obtained closed form, we have still found the generating function $F(z)$, which is arguably more useful than f_n.

4 Conclusion

What we investigated is only a small part of the power of the snake oil method. In conclusion, this sub-method is very useful whenever you have a sum which involves the expression $n - k$. The snake oil method has much broader applications, but even with just a part of it, we can solve many many summations. In future research, we will look at other variations of the snake oil method, and how to apply them.