E489: On the Unraveling of Exponential Formulas

L. Euler, Varun Srivastava

April 1, 2018

1 I

Euler begins by explaining that expressions of the form

$$r^{r^{r^{r^{\alpha}}}}$$

are poorly understood, and thus some techniques must be used to determine the basic properties of these expressions.

2 II

We begin by making the substitution $r^{\alpha} = \beta$ such that $r^{r^{\alpha}} = r^{\beta}$. He also lets $r^{\beta} = \gamma$, $r^{\gamma} = \epsilon$, $r^{\epsilon} = \zeta$, and so on. it is immediately clear that if we start from $\alpha = 0$, then $\beta = 1$, $\gamma = r$, $\delta = r^r$, $\epsilon = r^{r^r}$ and so on.

3 III

It is clear now that a larger value of r will yield a larger value of the expression. If we assume $r = 2$ and $\alpha = 0$, then $\beta = 1$, $\gamma = 2$, $\delta = 4$, $\epsilon = 16$, and so on. It is already clear that for sufficiently large of r, the expression will tend to infinity.

4 IV

If we begin by letting r be $\sqrt{2}$ and $\alpha = 2$, then $\beta = \sqrt{2}^2 = 2$, $\gamma = \sqrt{2}^2 = 2$, and so on.
If $\alpha = 4$, then $\beta = 4, \gamma = 4$, and so on.
If $\alpha = 6$, then $\beta = 8$, $\gamma = 16$, $\delta = 256$ and so on.

5 V

We will concern ourselves with finding the first value of r in which the expression will tend to infinity. This can be established in terms of logarithms. We know that $\log(\beta) = \alpha \log(r)$. It follows that $\log(\log(\beta)) = \log(\alpha) + \log(\log(r))$, $\log(\log(\gamma)) = \log(\beta) + \log(\log(r))$, $\log(\log(\gamma)) = \log(\delta) + \log(\log(r))$ and so on.
We begin by letting $r = \frac{3}{2}$, which for Euler very helpfully calculates that

$$\log(r) = 0.1760913 \text{ and } \log(\log(r)) = 9.2457379.$$

Euler omits mentioning that these logarithms are in base 10 and are calculated modulo 10. Hence why $\log(\log(r)) = 9.2457379$. Through addition and exponentiation by hand, we begin with $\alpha = 1.5000$ and calculate $\beta = 1.8371$, $\gamma = 2.1062$, $\delta = 2.3490$, $\epsilon = 2.5920$, $\zeta = 2.8604$, $\eta = 3.1893$, $\theta = 3.6443$.

6 VI

We repeat the same process as last time except by taking by taking both r and α to be $\frac{4}{3}$. Euler calculates $\log(\frac{4}{3}) = 0.1249387$ and $\log(\log(\frac{4}{3})) = 9.0966972$. This yields that $\beta = 1.4675$, $\gamma = 1.5252$, $\delta = 1.5508$, $\epsilon = 1.5622$, $\zeta = 1.5674$. Euler notes that the difference between terms is decreasing. Euler then states we can "safely" conclude that the expression must be bounded.

7 VII Problem:

What is the maximal value of r such that the sequence is bounded?

We can begin by setting the sequence equal to r^ω. Because the sequence "flattens," raising it to another power of r will not change anything. Thus $r^\omega = \omega$. Therefore the problem becomes finding the maximal value of r for our new expression.

8 VIII

We can begin by using logarithms. Because $\omega \log(r) = \omega$, we then can say $\log(r) = \frac{\log(\omega)}{\omega}$, or $r = \omega^{\frac{1}{\omega}}$. We take the derivative of $r = \frac{\log(\omega)}{\omega}$ relative to ω to yield $r' = \frac{1-\omega}{\omega^2}$. Setting r' to zero in order to find the maxima yields that $\ln(\omega) = 1$, or $\omega = e$. Therefore, the maximal value of r is $r = e^{\frac{1}{e}}$.

9 IX

We begin to approximate $e^{\frac{1}{e}}$ by calculating values near to it. Euler makes a table and calculates $2^{\frac{1}{2}} = 1.41421$ and $3^{\frac{1}{3}} = 1.44225$.

10 X

We let $z = e^{\frac{1}{e}}$. Working in base 10 and modulo 10, Euler calculates $\log(e) = 0.4342944$ and $\log(\log(e)) = 0.4342944$. Thus, he determines $z = 1.44467$. We note that this is approximately $1\frac{4}{9}$.

11 XI

Using the Maclaurin Expansion of e^x

$$e^x = 1 + x + \frac{1}{2}xx + \frac{1}{6}x^3\frac{1}{24}x^4 + \cdots$$

we observe that

$$e^{\frac{1}{e}} = 1 + \frac{1}{e} + \frac{1}{2ee} + \frac{1}{6e^3} + \frac{1}{24e^4} + \cdots .$$