E695

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Abstract

In this paper, Euler calculated the titular integral using a series of substitutions, as is his standard style. He does this in a more natural way than it was calculated prior to his paper.

1 Introduction to the Integral

The integral that Euler solved in this paper was

$$\int \frac{dz}{(3 \pm z^2)(\sqrt[3]{1 \pm 3z^2})}$$

. We define two informal sets of substitutions in the next two subsections.

1.1 z and v

Euler uses the variable v in terms of z. He writes $v = \sqrt[3]{1 \pm 3z^2}$. Then, $v^3 = 1 + 3z^2$ so $z^2 = \frac{v^3-1}{3}$. Taking the derivative, we get that $2z = v^2 dv$. Once more taking the derivative, we find $zdz = \frac{v^2 dv}{2}$. So,

$$dz = \frac{v^2 dv}{2z}$$

Euler also defines $dV = \frac{dz}{(3+z^2)(\sqrt[3]{1 \pm 3z^2})}$. Putting the numerator in terms of v, we find

$$dV = \frac{vdv}{2z(3 + z^2 2)}$$

1.2 p and q

Euler also extensively used the variables p and q. They are defined as $p = \frac{1+z}{v}$ and $q = \frac{1-z}{v}$. So, $p^3 + q^3 = 2$ and $p^3 - q^3 = \frac{6z+2z^3}{v^3}$. Also, $p + q = \frac{2}{v}$ and then $dp + dq = \frac{-2dv}{v^2}$. Combining the p, q, and dV, we find

$$dV = \frac{dp + dq}{2(p^3 - q^3)}$$

Euler also defines $dP = \frac{dp}{p^3 - q^3}$ and $dQ = \frac{dq}{p^3 - q^3}$. Then,

$$dV = (\frac{-1}{2})(dP + dQ)$$

Since $p^3 = 2 - q^3$, $dQ = \frac{dq}{2(1-q^3)}$ so

$$4dV = \frac{dp}{1 - p^3} - \frac{dq}{1 - q^3}$$

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2 Solving using the Substitutions

It is generally accepted and widely used at Euler's time that

$$\int \frac{dp}{1-p^3} = (\frac{1}{3}) \ln \frac{\sqrt[2]{1+p+p^2}}{1-p} + \frac{1}{\sqrt[2]{3}} \arctan \frac{p\sqrt[2]{3}}{2+p}$$

We substitute $1+p+p^2 = \frac{1-p^3}{1-p}$. Then, $(\frac{1+p+p^2}{(1-p)^2})^{1/2} = (\frac{1-p^3}{(1-p)^3})^{1/2}$. So, $(\frac{1}{3}) \ln \frac{\sqrt[2]{1+p+p^2}}{1-p} = (\frac{1}{6}) \ln \frac{1-p^3}{(1-p)^3}$.

Similarly,

$$\int \frac{dq}{1-q^3} = (\frac{1}{6}) \ln \frac{1-q^3}{(1-q)^3} + \frac{1}{\sqrt[2]{3}} \arctan \frac{q\sqrt[2]{3}}{2+q}$$

Then, going back, we fint

$$\int 4dV = \int \frac{dp}{1-p^3} - \frac{dq}{1-q^3}$$

Combining our substitutions and derivations, we get

$$4V = (\frac{1}{6}) \ln \frac{1-p^3}{(1-p)^3} - (\frac{1}{6}) \ln \frac{1-q^3}{(1-q)^3} + \frac{1}{\sqrt[2]{3}} \arctan \frac{p\sqrt[2]{3}}{2+p} - \frac{1}{\sqrt[2]{3}} \arctan \frac{q\sqrt[2]{3}}{2+q}$$

Now combining the logarithms, we find that $(\frac{1}{6}) \ln \frac{1-p^3}{(1-p)^3} - (\frac{1}{6}) \ln \frac{1-q^3}{(1-q)^3} = (\frac{1}{6}) \ln \frac{(1-p^3)(1-q^3)}{((1-p)^3)((1-q)^3)} = (\frac{1}{6}) \ln \frac{1-p^3}{1-q^3} + (\frac{1}{6}) \ln \frac{(1-p)^3}{(1-q)^3} = (\frac{1}{6}) \ln(-1) + (\frac{1}{6}) \ln \frac{1-p^3}{(1-q)^3}$. Also, $1-p^3 = -(1-q^3)$, so $(\frac{1}{6}) \ln \frac{(1-p)^3}{(1-q)^3} = (\frac{1}{2}) \ln \frac{1-q}{1-p}$.

Euler treats indeterminate complex quantities as constants, and since $(\frac{1}{6}) \ln(-1)$ is like a constant, Euler eliminates it from his final solution.

Therefore,

$$4V = (\frac{1}{2}) \ln \frac{1-q}{1-p} + \frac{1}{\sqrt[2]{3}} \arctan \frac{p\sqrt[2]{3}}{2+p} - \frac{1}{\sqrt[2]{3}} \arctan \frac{q\sqrt[2]{3}}{2+q}$$

This gives us

$$V = (\frac{1}{8}) \ln \frac{1-q}{1-p} + \frac{1}{4\sqrt[2]{3}} \arctan \frac{(p-q)\sqrt[2]{3}}{2+p+q+2pq}$$

We finally find, where $v = \sqrt[3]{1+3z^2}$,l

$$\int \frac{dz}{(3 \pm z^2)(\sqrt[3]{1 \pm 3z^2})} = (\frac{1}{8}) \ln \frac{1-v-z}{1-v+z} + \frac{1}{4\sqrt[2]{3}} \arctan \frac{vz\sqrt[2]{3}}{1+v+v^2-z^2}$$

Link: http://eulerarchive.maa.org/pages/E695.html