Elliptic Integrals
Investigations in E448

L. Euler, Summarized by: Ed Chen

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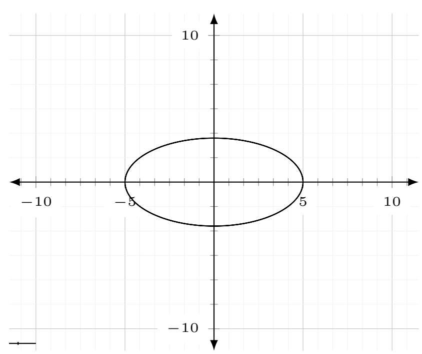
Leohnard Euler maintains himself as one of the great mathematicians to have ever lived. Throughout his work, we see he has pioneered a multitude of mathematical sciences and fields, including astronomy, number theory, differential calculus, upon many more. Here, we seek to look into his specific papers denoted by their Enestrom number, E448 to analyze some of his work on Elliptic Integrals, 30 years apart.

To start off, we will look into Euler's attempts to evaluate a good approximation of the arc length of an ellipse, and explore different more modern methods to accomplish the same task using simpler mathematics. In his paper E448 he develops a series that is shown to quickly converge and provides the standard approximation for a ellipse's perimeter that is generally used today. Euler starts off by saying that numerous mathematicians had concluded that there was no simple way to express the perimeter of an ellipse.

1 E448 Rectification of an Ellipse

§1 First, assume the basic equation for an ellipse with semi axises b and a

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Euler provides the substitution

$$\frac{x^2}{a^2} = \frac{1+z}{2} \qquad \frac{y^2}{b^2} = \frac{1-z}{2}$$

Using this, we seek to find the formula for arc length using these substitutions or $\int ds$ where ds in this case is in terms of this parametric variable z, or in other words $ds = \sqrt{\frac{dx}{dz}^2 + \frac{dy}{dz}^2}$ Solving, we get

$$dx = \frac{adz}{2\sqrt{2(1+z)}} \qquad dy = \frac{-bdz}{2\sqrt{2(1-z)}}$$

which, with a little manipulation yields

$$s = \frac{1}{2\sqrt{2}} \int_{-1}^{1} \sqrt{\frac{a^2 + b^2 - (a^2 - b^2)z}{1 - z^2}} dz$$

where, when the parametric is evaluated from the bounds -1 to 1, and the constant is factored out, yields the perimeter of the whole ellipse.

§2 Following this, Euler defines 2 new variables c and n as

$$c^2 = a^2 + b^2 \qquad n = \frac{a^2 - b^2}{a^2 + b^2}$$

which, when substituted back into the formula for s, yields

$$s = \frac{c}{2\sqrt{2}} \int_{-1}^{1} \frac{\sqrt{1 - nz}}{\sqrt{1 - z^2}} dz$$

Through the binomial formula, $(1 + x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3...$ where $k = \frac{1}{2}$ and $x = nz$ we see that

$$\sqrt{1 - nz} = 1 - \frac{1}{2}nz - \frac{1 \cdot 1}{2 \cdot 4}n^2 z^2 - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6}n^3 z^3$$

When we plug in this expanded formula into the integral, we find that the fraction and the power of n can be factored out, leaving just the powers of z Integrating a few of the first couple terms yields

$$\int_{-1}^{1} \frac{dz}{\sqrt{1 - z^2}} = \pi \qquad \int_{-1}^{1} \frac{zdz}{\sqrt{1 - z^2}} = 0$$

A bit of further investigation, we find that we can evaluate all the other values of this integral at any power of z given by λ through using the 2 values given above and the formula

$$\int_{-1}^{1} \frac{z^{\lambda+2}}{\sqrt{1 - z^2}} dz = \frac{\lambda + 1}{\lambda + 2} \int_{-1}^{1} \frac{z^{\lambda}}{\sqrt{1 - z^2}} dz$$

§3 Using the above formula, we see that all odd powers of z would yield 0 so we only need to consider the even powers of z. Evaluating with the above formula, we get that the first few iterations of the even powers of yields z,

$$\int_{-1}^{1} \frac{dz}{\sqrt{1 - z^2}} = \pi \qquad \int_{-1}^{1} \frac{z^2 dz}{\sqrt{1 - z^2}} = \frac{1}{2}\pi$$

$$\int_{-1}^{1} \frac{z^4 dz}{\sqrt{1 - z^2}} = \frac{1 \cdot 3}{2 \cdot 4}\pi \qquad \int_{-1}^{1} \frac{z^6 dz}{\sqrt{1 - z^2}} = \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\pi$$

Evidently, each term carries a π so we can factor that out. From the last simplified formula for s, if we plug in the constant coefficients in both the expansion of the binomial formula, and the integrals above

$$s = \frac{c\pi}{2\sqrt{2}}(1 - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{1}{2}n^2 - \frac{1 \cdot 1}{4 \cdot 4} \cdot \frac{3 \cdot 5}{8 \cdot 8}n^4...)$$

when you simplify with the previously defined variables c and n the sequence yields the infinite series for the perimeter of an ellipse, which provides a relatively good approximation, even with only the first 5 terms converges rather quickly. For instance, if we take only the first term we get the popular approximation for the perimeter of an ellipse

$$\pi\sqrt{2(a^2 + b^2)}$$

For instance, if we use $a = 5$ and $b = 4$ then we get 28.36.

2 Alternate way to derive Perimeter of an Ellipse

As previously stated, we are trying to find the perimeter of the ellipse shown above, with the semi-axises b & a. First, take the parametric equations for the equation of an ellipse

$$x = acos(\theta) \qquad y = bsin(\theta)$$

Using the formula for arc length,

$$\int ds = \int \sqrt{(\frac{dx}{d\theta})^2 + (\frac{dy}{d\theta})^2} d\theta$$

$$p = 4\int_0^{\frac{\pi}{2}} \sqrt{a^2 sin^2\theta + b^2 cost^2\theta} d\theta$$

which when simplified and using $\varepsilon = \sqrt{1 = \frac{b^2}{a^2}}$ we get

$$4a\int_0^{\frac{\pi}{2}} \sqrt{1 - \varepsilon^2 cos^2\theta} d\theta$$

This, is what we know as an elliptical integral, which does not have an established way to be evaluated using simple functions. First, notice that using the binomial formula, we get denote that

$$\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n+1} 1 \cdot 3 \cdot 5 ... (2n-3)x^n}{2^n n!}$$

Through setting x equal to the corresponding part in the above elliptical integral, we can express this root as

$$\sqrt{1 - \varepsilon^2 cos^2\theta} = 1 - \frac{\varepsilon^2 cos^2\theta}{2} - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 ... (2n-3)\varepsilon^{2n} cos^{2n}\theta}{2^n n!}$$

Notice that since $cos\theta$ is bound between 0 and 1, this series always converges for all values of θ Using the previously established formula for the elliptical integral, we can derive that

$$p = 4a\int_0^{\frac{\pi}{2}} 1 - \frac{\varepsilon^2 cos^2\theta}{2} - \sum_{n=2}^{\infty} \frac{1 \cdot 3 \cdot 5 ... (2n-3)\varepsilon^{2n} cos^{2n}\theta}{2^n n!} d\theta$$

integrating everything termwise, we get

$$p = 4a\left(\frac{\pi}{2} - \frac{\varepsilon^2}{2}(\frac{1}{2} \cdot \frac{\pi}{2}) - \sum_{n=2}^{\infty} \frac{(1 \cdot 3 \cdot 5 ... (2n-3)\varepsilon^{2n})^2}{(2^n n!)^2 (2n-1)} \cdot \frac{\pi}{2}\right)$$

which when simplified, also yields the formula

$$p = 2\pi a\left(1 - (\frac{1}{2})^2 \frac{\varepsilon^2}{1} - (\frac{1 \cdot 3}{2 \cdot 4})^2 \frac{\varepsilon^4}{3} ...\right)$$

yielding the same result as Euler's derivation.