THREE SQUARES

DRAFT - ARJUN

1. Background

Euler initially was searching for numbers x, y, and z such that $x+y+z$, $xy+yz+zx$, and xyz were all squares, before settling on the subcase where x, y, and z were all squares. In this scenario, xyz is automatically a square, and we can rewrite the condition: we want to find x, y, and z such that

$$x^2 + y^2 + z^2 = P^2$$

$$x^2y^2 + y^2z^2 + z^2x^2 = Q^2$$

First, we can search for a nice form for x, y, and z satisfying the first condition. We are initially reminded of the generating formula for Pythagorean triples:

$$\begin{aligned} a &= m^2 = n^2 \\ b &= 2mn \\ c &= m^2 + n^2 \end{aligned}$$

.

We can emulate this formula here as follows:

$$\begin{aligned} x &= p^2 + q^2 - r^2 \\ y &= 2pr \\ z &= 2qr \\ P &= p^2 + q^2 + r^2 \end{aligned}$$

.

We can now plug this form into the second condition:

$$x^2(y^2 + z^2) + y^2z^2 = Q^2$$

$$(p^2 + q^2 - r^2)^2 4r^2(p^2 + q^2) + 16p^2q^2r^4 = Q^2$$

$$\frac{Q^2}{4r^2} = (p^2 + q^2 - r^2)^2(p^2 + q^2) + 4p^2q^2r^2$$

Thus the right hand side must also be a square.

However, from this equation, the only general solution that can be found occurs when $r^2 = p^2 + q^2$, which yields $x = 0$ while (y, z, P) is some Pythagorean triple.

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2. Substituting for r

Euler then picks the ingenious substitution $r = p - nq$, yielding

$$\frac{Q^2}{4(p-nq)^2} = (q^2 + 2pnq - n^2q^2)^2(p^2+q^2) + 4p^2q^2(p-nq)^2$$
$$\frac{Q^2}{4q(p-nq)^2} = (2pn + (1-n^2)q)^2(p^2+q^2) + 4p^2(p-nq)^2$$

Because the left side of this equation must be a square, set it as R^2. Expanding the right side gives

$$R^2 = 4(n^2+1)p^4 - 4n(1+n^2)p^3q + (1+6n^2+n^4)p^2q^2 + 4n(1-n^2)pq^3 + (1-n^2)^2q^4$$

For this to be the square of a quadratic, the first and last terms must be squares. In this case, the last term is a square, so we can use it along with the third and fourth terms to create a new set of conditions.

Taking $R = \alpha p^2 + 2npq + (1-n^2)q^2$ gives

$$R^2 = \alpha^2p^4 + 4n\alpha p^3q + (4n^2 + 2\alpha(1-n^2))p^2q^2 + 4n(1-n^2)pq^3 + (1-n^2)^2q^4$$

with the final two terms equal to those of the original representation. The middle terms are equated when $4n^2 + 2\alpha(1-n^2) = (1+6n^2+n^4)$, or $\alpha = \frac{(1+n^2)^2}{2(1-n^2)}$.

We can now equate these two representations of R^2; the final three terms cancel, yielding

$$4(n^2+1)p^4 - 4n(1+n^2)p^3q = \alpha^2p^4 + 4n\alpha p^3q = \frac{(1+n^2)^4}{4(1-n^2)^2}p^4 + \frac{2n(1+n^2)^2}{1-n^2}p^3q$$

$$4p^4 - 4np^3q = \frac{(1+n^2)^3}{4(1-n^2)^2}p^4 + \frac{2n(1+n^2)}{1-n^2}p^3q$$

$$4(1-n^2)^2(4p^4 - 4np^3q) = (1+n^2)^3p^4 + 8n(1-n^4)p^3q$$

$$4(1-n^2)^2(4p - 4nq) = (1+n^2)^3p + 8n(1-n^4)q$$

$$(16 - 32n^2 + 16n^4)p - (16n - 32n^3 + 16n^5)q = (1 + 3n^2 + 3n^4 + n^6)p + (8n - 8n^5)q$$

$$(15 - 35n^2 + 13n^4 - n^6)p = (24n - 32n^3 + 8n^5)q$$

$$(n^4 - 10n^2 + 5)p = 8n(1-n^2)q$$

$$\frac{p}{q} = \frac{8n(1-n^2)}{(n^4 - 10n^2 + 5)}$$

The easiest way to find solutions from this is to set p and q directly equal to the numerator and denominator on the right side (and then scale afterward).

We then have

$$p = 8n(1 - n^2)$$
$$q = n^4 - 10n^2 + 5$$
$$r = p - nq = n(3 + 2n^2 - n^4)$$

.

These representations, along with the formulas

$$x = (p^2 + q^2 - r^2)$$
$$y = 2pr$$
$$z = 2qr$$

allow us to find triples that satisfy these equations by plugging in various values of n

3. Examples

for $n = 2$, we get

$$p = 8n(1 - n^2) = -48$$
$$q = n^4 - 10n^2 + 5 = -19$$
$$r = p - 2q = -10$$

$$x = (p^2 + q^2 - r^2) = 2565$$
$$y = 2pr = 960$$
$$z = 2qr = 380$$

Because we can scale, dividing by 5 gives $(x, y, z) = (513, 192, 76)$ From this, we then get $P = 553$, $Q = 106932$.

For $n = 3$, we get $(p, q, r) = (-192, -4, -180)$, which can be reduced to $(48, 1, 45)$. From here we get $(x, y, z) = (280, 4320, 90)$ which in turn can be reduced to $(28, 432, 9)$, which is the lowest solution we can obtain using this method.

4. Introducing m

To go further, we take the case where $n^2 + 1$ is a square, and set m as its square root.

We then get the following equation:

$$R^2 = 4m^2p^4 - 4nm^2p^3q + (m^4 + 4n^2)p^2q^2 + 4n(1 - n^2)pq^3 + (1 - n^2)^2q^4$$

Now, in addition to using the last three terms, we can also use other combinations:

4.1. First, Second, and Last Terms.

Taking the first, second, and last terms, we get $R = 2mp^2 - mnpq + (1 - n^2)q^2$.

Equating the two forms of R^2 gives us

$$\frac{p}{q} = \frac{2n(4 + 2m - 2m^2 - m^3}{4 + 8m - 5m^2 - 4m^3}$$

Setting $a = 2$, $b = 1$ gives the final values for $196, 693, 528$, which are much larger than the values from the previous method

4.2. **First, Fourth, and Last Terms.** Taking the first, fourth, and last terms, we get $R = 2mp^2 + 2npq + (1 - n^2)q^2$.

Equating and moving terms gives

$$\frac{p}{q} = \frac{4m - 4mn^2 - m^4}{4mn(2+m)}$$

which simplifies to

$$\frac{4 - 2m - m^2}{4n}$$

Plugging in $a = 2$, $b = 1$ gives values $108, 7, 336$ for x, y, z. Here P is larger than in the above case, and Q is smaller.

4.3. **First Three Terms.** Equating the first three terms gives us $R = 2mp^2 - mnpq + \frac{m^4+3n^2}{4m}q^2$

From which we get

$$\frac{p}{q} = \frac{m^8 - 2m^4n^2(n^2-3) + n^4(n^2-3) - 16m^2(n^2-1)^2}{8 + m^4 - 5n^2 - n^4}$$

There is no simpler result that can be obtained from this equation. Thus, these are the smallest values that can be extrapolated from this method.

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