E228: A SUMMARY OF 'ON NUMBERS WHICH ARE THE SUM OF TWO SQUARES'

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1. A few simple observations

(1) Even squares are of the form $4a$
(2) Odd squares are of the form $8b + 1$
(3) Sum of 2 even squares is of the form $4a$
(4) Sum of 1 even and 1 odd square $4a + 1$
(5) Sum of 2 odd squares: $8a + 2$
(6) Odd numbers are either $4n - 1$ or $4n + 1$
(7) Those that are of the form $4n - 1$ are not $a^2 + b^2$
(8) Not all numbers of the form $4n + 1$ or $8n + 2$ are sums of squares. For example, $21, 33, 57, \ldots$ and $42, 66, 77, \ldots$
(9) A few "well known" lemmas:
   (a) If $p = a^2 + b^2$, $n^2p = c^2 + d^2$
   (b) If $p = a^2 + b^2$, $2p = (a + b)^2 + (a - b)^2$
   (c) If $2p = a^2 + b^2$, $p = (\frac{a+b}{2})^2 + (\frac{a-b}{2})^2$

2. Theorems

Theorem 1. *If $p = a^2 + b^2$ and $q = c^2 + d^2$ then $pq = j^2 + k^2$.*

Proof. We proceed as follows:

$$pq = (a^2 + b^2)(c^2 + d^2)$$

□

$$pq = (ac + bd)^2 + (ad - bc)^2$$

The converse is not true, but if $pq = a^2 + b^2$ and $p = c^2 + d^2$ then $q = i^2 + j^2$. For a proof of this, see Rajiv's paper.

Proposition 2. *If $pq = a^2 + b^2$ then p is prime and $p = c^2 + d^2$ and $q = i^2 + j^2$.*

Proof. We proceed as folows:

$$pq = a^2 + b^2$$

$$p = c^2 + d^2$$

Since p is prime, $g.c.d(c, d) = 1$.

$$q = \frac{a^2 + b^2}{c^2 + d^2}$$

$$c^2 + d^2 | c^2(a^2 + b^2)$$

$$c^2 + d^2 | a^2(c^2 + d^2)$$

$$c^2 + d^2 | b^2c^2 - a^2d^2$$

$$c^2 + d^2 | (bc + ad)(bc - ad)$$

$$c^2 + d^2 | bc \pm ad$$

For the purpose of brevity we will illustrate only the case where $c^2+d^2|bc+ad$, but the reader may work out the other similarly. Let $bc+ad = mc^2+md^2$, $b = mc+x$, $a = md + y$.

$$mc^2 + xc + md^2 + dy = mc^2 + md^2$$

$$xc + dy = 0$$

$$\frac{x}{y} = -\frac{d}{c}$$

$$gcd(d, c) = 1$$

$$x = nd$$

$$y = -nc$$

$$a = md - nc$$

$$b = mc + nd$$

$$pq = m^2d^2 - 2mncd + n^2c^2 + m^2c^2 + 2mncd + n^2d^2$$

$$pq = (m^2 = n^2)(c^2 + d^2)$$

□