The Complex Exponential Function

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1 Introduction

The obvious first question is what is the complex exponential function. It was discovered by Euler and turns out to be exactly what its name suggests- an exponential function for complex numbers. That is, it is e^z where z is some complex number.

2 Euler's Formula

The most common example of the importance or usefulness of the complex exponential function comes in the form of Euler's formula. It is:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

This property is often used in many contexts and also helps make complex numbers easier to multiply and divide as we will later see. However, first let us consider some of the proofs for Euler's formula.

2.1 Taylor Series's Proof

This proof is one of the most well known due to it being Euler's proof. For this proof, we recall that a Taylor series centered around 0, also known as a Maclaurin Series, is nothing but

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0)$$

We can write our the Taylor expansion for the exponential function.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

If we let x be iy, we have:

$$e^{iy} = 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + ...$$

We know that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, and so on. As a result, our terms either are a multiple of i or not. We can separate e^{iy} into our real and imaginary parts.

$$e^{iy} = (1 - \frac{y^2}{2!} + \frac{y^4}{4!} - ...) + i(y - \frac{y^3}{3!} + \frac{y^5}{5!} - ...)$$

Now we can note that the real part of this series is nothing but the Taylor expansion for $\cos(y)$ and the imaginary part is just i times the Taylor expansion for $\sin(y)$. So that means that

$$e^{iy} = \cos(y) + i\sin(y)$$

our desired equation and we have completed our first proof for Euler's formula.

2.2 Proof by Constant Derivative

In this proof we start with the following function of θ:

$$f(\theta) = e^{-i\theta}(\cos(\theta) + i\sin(\theta))$$

If we are able to show that this function is always equal to 1, we will be able to prove Euler's formula. We can start to show this by taking the derivative of $f(\theta)$ to get:

$$f'(\theta) = -ie^{-i\theta}(\cos(\theta) + i\sin(\theta)) + e^{-i\theta}(-\sin(\theta) + i\cos(\theta))$$

All the terms on the right hand side cancel out, leaving 0. That is, $f'(\theta) = 0$. If the derivative of the function is 0, we know that the function itself must equal a constant. That is, $f(\theta) = k$ for some constant k for all θ. Since this is true for all θ, we can find the function value for any θ and that will be our function value for all θ. Letting $\theta = 0$, we have:

$$f(0) = e^{-i*0}(\cos(0) + i\sin(0) = 1$$

Therefore, we have that in general:

$$f(\theta) = e^{-i\theta}(\cos(\theta) + i\sin(\theta)) = 1$$

Rearranging the latter half of the equations, we have:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

And we have completed our second proof for Euler's formula and only our third proof remains.

2.3 Proof via Differential Equation

For this proof, we can start by trying to relate a function to its derivative in order to solve the resulting differential equation to find an equivalent function. Let us start with the following function:

$$y = \cos(x) + i\sin(x)$$

We can take the derivative of both sides to learn that

$$\frac{dy}{dx} = -\sin(x) + i\cos(x)$$

We can note that the expression for the derivative, is nothing but:

$$\frac{dy}{dx} = -\sin(x) + i\cos(x) = i(\cos(x) + i\sin(x)) = iy$$

This creates our differential equation:

$$\frac{dy}{dx} = iy$$

While for those who have worked with DEs often, the solution may become readily apparent, we can also separate the variables to solve this equation. By rearranging, we have:

$$\frac{1}{y}dy = idx$$

Taking the integral of both sides, we obtain

$$\ln(y) = ix$$

By raising both sides to e, we have that $y = e^{ix}$ and equating the expressions we have for y, we have:

$$e^{iy} = \cos(y) + i\sin(y)$$

completing our final proof for Euler's formula.

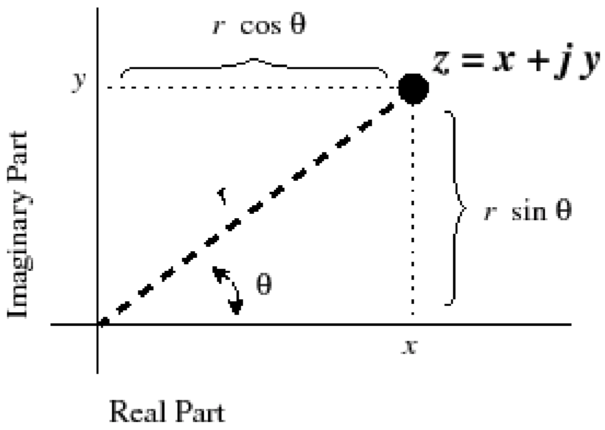

Figure 1: This complex plane graph shows how we can write the real and imaginary parts of a complex numbers in terms of the angle theta and the magnitude of z (i.e.|z|, also written as r).

3 Interesting Resulting Properties

The complex exponential function makes it much easier to work with complex numbers, to find roots of equations, and to derive trigonometric identities.

3.1 Complex Numbers

To start, consider a complex number $z = x + iy$. If we draw it on the complex plane as in Figure 1, we can let theta be the angle z makes with the real axis or x-axis. Projecting z on to the real and imaginary axis, we have that $x = |z|\cos(\theta)$ and $y = |z|\sin(\theta)$ where $|z|$ is the magnitude of the complex number, or $\sqrt{x^2 + y^2}$. Then we can write z as:

$$z = x + iy = |z|\cos(\theta) + i|z|\sin(\theta) = |z|e^{i\theta}$$

by using Euler's formula on that last step. This allows us to write every complex number as a constant times complex exponential function. This readily reveals several tricks for dealing with complex numbers including allowing for easier multiplication and division of complex number.

3.2 Roots

This property of complex numbers which allow them to be written as a magnitude times an exponential function also makes it easy to find roots. This is because the nth power of a complex number can be written as:

$$z^n = r^n e^{in\theta}$$

Therefore, say we are trying to find the roots of $z^3 = 1$. We can write this as:

$$z^3 = r^3 e^{3i\theta} = 1e^{0i}$$

This easily reveals that our solutions have r = 1 and $3i\theta = 0 + 2\pi k$ With k = 0,1,2 we get our three unique solutions for $\theta = \frac{2\pi k}{3}$ Similarly, we have our solutions for any equation $z^n = 1$ to be:

$$z = e^{\frac{2\pi k}{n} i}$$

for k = 0, 1, 2, ..., n-1. This is a nice compact formula for roots of 1.

3.3 Trigonometric Identities

Due to the properties of exponential functions. We also know that:

$$(e^{i\theta})^n = e^{i\theta n}$$

This is also known as DeMoivre's formula. From Euler's formula, we can write this expression as:

$$(\cos(\theta) + i\sin(\theta))^n = \cos(n\theta) + i\sin(n\theta)$$

This is what helps us derive trigonometric identities. For example, let n = 2. Then we have:

$$(\cos(\theta) + i\sin(\theta))^2 = \cos^2(\theta) - \sin^2(\theta) + 2i\cos(\theta)\sin(\theta)$$

We know that this is just $\cos(2\theta) + i\sin(2\theta)$. So by matching the real and imaginary parts. We have that:

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$

$$\sin(2\theta) = 2\cos(\theta)\sin(\theta)$$

Similarly, we can do this for n = 3. From the first part, we have:

$$(\cos(\theta) + i\sin(\theta))^3 = \cos(\theta)^3 + 3i\sin(\theta)\cos^2(\theta) - 3\cos(\theta)\sin^2(\theta) - i\sin^3(\theta)$$

Dividing the expression into its real and imaginary parts we have:

$$\cos(3\theta) + i\sin(3\theta) = [\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta)] + i[3\sin(\theta)\cos^2(\theta) - \sin^3(\theta)]$$

$$= [\cos^3(\theta) - 3\cos(\theta)(1 - \cos^2(\theta))] + i[3\sin(\theta)(1 - \sin^2(\theta)) - \sin^3(\theta)]$$

$$= [4\cos^3(\theta) - 3\cos(\theta)] + i[3\sin(\theta) - 4\sin^3(\theta)]$$

So we have our identities:

$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$$

$$\sin(3\theta) = 3\sin(\theta) - 4\sin^3(\theta)$$

In this fashion, we can obtain trigonometric identities for any $\cos(n\theta)$ and $\sin(n\theta)$.

4 Fourier Series

We conclude with a section on Fourier Series. The idea behind Fourier series is to map any periodic function in terms of sine and cosine. A reason for why we may want to do this lies in differential equations and the fact that sine and cosine are nice functions when working and solving differential equations. An interesting side note and explanation of what Fourier series represent lies in the fact that any time-dependent closed path can be emulated by infinitely many circles of different frequencies. The radii of those circles form a Fourier series.

That said, we can try to generate the form of a Fourier series for a periodic function. For the sake of simplicity, let us start with a periodic function with period of 2π. Then our function can be written as follows:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$

With some start constant term a_0 and weighted terms a_n for functions of cosine of period 2π and every smaller half of that period (i.e. $\pi, \frac{\pi}{2}, ...$) and weighted terms b_n for functions of sine of equivalent periods. With this, we can try to make to make it complex with the use of the complex exponential function. Start by noting that,

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \quad \text{and} \quad \sin(t) = \frac{e^{it} - e^{-it}}{2i}$$

So our function in the Fourier form can be written as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n (\frac{e^{it} + e^{-it}}{2}) + \sum_{n=1}^{\infty} b_n (\frac{e^{it} - e^{-it}}{2i})$$

Rearranging, we have that:

$$f(t) = a_0 + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{int} + \sum_{n=1}^{\infty} \frac{a_n + ib_n}{2} e^{-int}$$

We can simplify and say that:

$$f(t) = \sum_{n=1}^{\infty} c_n e^{int}$$

where $c_n = a_0$ when $n = 0$, $c_n = \frac{a_n - ib_n}{2}$ when $n = 1, 2, 3, ...$, and $c_n = \frac{a_n + ib_n}{2}$ when $n = -1, -2, -3, ...$ Now the question remains- what are a_n, b_n, and c_n. We know that c_n can be found from a_n and b_n so let us start by determining those. Remember the original form:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nt) + \sum_{n=1}^{\infty} b_n \sin(nt)$$

We can notice that we can determine the value of a_n or b_n where n is not 0 by multiplying the Fourier series' equation by the function for which the a_n or b_n is the weight and then taking the integral of both sides from $-\pi$ to π. In this way, all terms on the right side of the equation will go to zero with the integral except for the term $a_n \cos(nt)\cos(nt)$ or $b_n \sin(nt)\sin(nt)$ whose integral in being taken will go to $a_n\pi$ or $b_n\pi$ respectively. Therefore we are left with:

$$\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = a_n\pi \qquad \text{or} \qquad \int_{-\pi}^{\pi} f(t)\sin(nt)\,dt = b_n\pi$$

Solving for a_n and b_n we have:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt \qquad \text{and} \qquad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt$$

For a_0, we can just take the integral of both sides of the equations, leaving that:

$$\int_{-\pi}^{\pi} f(t)\,dt = \int_{-\pi}^{\pi} a_0\,dt$$

and rearranging that,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$$

We can now return back to evaluate c_n. For n > 0, we have:

$$c_n = \frac{a_n - ib_n}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\cos(nt) - i\sin(nt)]f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}f(t)\,dt$$

If we let n < 0, we have that:

$$c_{-n} = \frac{a_{-n} + ib_{-n}}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi}[\cos(nt) - i\sin(nt)]f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}f(t)\,dt$$

Finally, we have that when n $= 0$:

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i*0*t}f(t)\,dt \ = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}f(t)\,dt$$

Therefore, for all c, we have that:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-int}f(t)\,dt$$

for our Fourier series

$$f(t) = \sum_{n=1}^{\infty} c_n e^{int}$$

The last thing we want to do is to extend this to functions of all periods and not just those of period 2π. We can easily note that if the function has period L, that we have that

$$c_n = \frac{1}{L}\int_{-L/2}^{L/2} e^{-int\frac{2\pi}{L}}f(t)\,dt$$

in the general Fourier series

$$f(t) = \sum_{n=1}^{\infty} c_n e^{-int\frac{2\pi}{L}}$$

We now have the ability to write all periodic functions in terms of the complex exponential function which is often easier to integrate than sine and cosine and will allow for easier solving for the constants of the Fourier series. In addition, with the single form for c_n, it is easier to work with complex Fourier Series then the previous form. As a result, Fourier Series provide yet another helpful application of the complex exponential function.