EULER 123: OBSERVATIONS ON CONTINUED FRACTIONS

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Let us begin with looking at ways to manipulate the continued fraction formula

$$A+\cfrac{B}{C+\cfrac{D}{E+\cfrac{F}{G+\cfrac{H}{I+...}}}}.(1)$$

Look at approximating the general formula. Chopping off bits one by one will get you alternating above and below approximations. If you then make them normal fractions, you get $\frac{A}{1}, \frac{AC+B}{C}, \frac{ACE+BE+DA}{CE+D}$... Sure, it seems rather random, but the rule is that you multiply the top and bottom by the next denominator thing (*C, *E, *G, ...), then then you add to the numerator the two previous times the next numerator thing, and add to the denominator the two previous times the next numerator thing. It's kind of clear to see basically how it works, and proving it isn't interesting, so let's continue. So now, if we take differences, to get an alternating infinite series, getting

$$A+\frac{B}{P}-\frac{BD}{PQ}+\frac{BDF}{QR}-\frac{BDFH}{RS}+etc.(2a)$$

, where

$$P=C, Q=EP+D, R=GQ+FP, S=IR+HQ, etc.(2b)$$

As we know that $C=P, E=\frac{Q-D}{P}, G=\frac{R-FP}{Q}, I=\frac{S-HQ}{R}$ by simple substitution in the above equations, we get $\frac{B}{P+\frac{D}{\frac{Q-D}{P}+\frac{F}{\frac{R-FD}{Q}+...}}}$, as it is clearly better to instead have infinite levels of fractions, but to have extra fractions on a level as well. However, even Euler isn't that crazy so you can multiply to get rid of the fractions, giving

$$\cfrac{B}{P+\cfrac{DP}{Q-D+\cfrac{FPQ}{S-HQ+\cfrac{KRS}{etc.}}}}.(3)$$

Now, we can let the series be $\frac{a}{p}-\frac{b}{q}+\frac{c}{r}-\frac{d}{s}+\frac{e}{t}, ...$ by substituting above $B=a, D=\frac{b}{a}, F=\frac{c}{b}, H=\frac{d}{c}, K=\frac{e}{d}, etc.$ and $P=p, Q=\frac{q}{p}, R=\frac{pr}{q}, S=\frac{qs}{pr}, T=\frac{prt}{qs}etc.$ Plugging that into the formula, to maximize ugliness, we get after getting rid of fractions

$$\cfrac{a}{p+\cfrac{bp^2}{aq-pb+\cfrac{acq^2}{br-cq+\cfrac{bdr^2}{cs-dr+\cfrac{ces^2}{dt-es+etc.}}}}}(4)$$

This just looks ugly. However, surprisingly, this has actual uses. For example, look at the alternating harmonic series. Now, this gives $a=b=c=...=1$, and $p=1, q=2, r=3, s=4$... Substituting that in, we get $\frac{1}{1+\frac{1}{1+\frac{4}{1+\frac{9}{1+\frac{16}{1+...}}}}}$. We get that by as the numerator is some 1's

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and then the next of the $p, q, r, s...$ sequence squared, and so the numerators are squares, and the $+'s$ are $1*n-1*(n-1)$, and so give you a 1 there. That means that as the alternating harmonic series is $\log 2$, we get that $\log 2 = \frac{1}{1+\frac{1}{1+\frac{4}{1+\frac{9}{1+\frac{16}{1+...}}}}}$. Another way, though. Let's start with the obvious statement, provable by induction, that

$$a^2 = \left(a-1+\frac{1}{2(a-1)+\frac{9}{2(a-1)+\frac{25}{2(a-1)...}}}\right)\cdot\left(a+1+\frac{1}{2(a+1)+\frac{9}{2(a+1)+\frac{25}{2(a+1)...}}}\right).(5)$$

Now

$$(a+2)^2 = \left(a+1+\frac{1}{2(a+1)+\frac{9}{2(a+1)+...}}\right)\cdot\left(a+3+\frac{1}{2(a+3)+\frac{9}{2(a+3)+...}}\right)$$

as well, simply by increasing a by two. Now, there is a common thing. So, if we look at $\frac{a^2}{(a+2)^2}$, we get

$$\frac{\left(a-1+\frac{1}{2(a-1)+\frac{9}{2(a-1)+...}}\right)}{\left(a+3+\frac{1}{2(a+3)+\frac{9}{2(a+3)+...}}\right)}.$$

However we can then multiply by $\frac{(a+4)^2}{(a+6)^2}$ and change those 3's in the denominator to 7's. And so on, and so on, and eventually the denominator is completely canceled out. So, we get that $\frac{a^2(a+4)^2(a+8)^2...}{(a+2)^2(a+6)^2(a+10)^2...} = \left(a-1+\frac{1}{2(a-1)+\frac{9}{2(a-1)+...}}\right)$. So we can plug in 2 to get that $\frac{2^2\cdot 6^2\cdot 10^2\cdot...}{4^2\cdot 8^2\cdot 12^2\cdot...} = \left(1+\frac{1}{2+\frac{9}{2+\frac{25}{2+...}}}\right)$. And canceling out 2^2 we get $\frac{1*1*3*3*5*5*7*7*9*9...}{2*2*4*4*6*6*8*8*...}$ on the left, and we already know that the right side is $\frac{4}{\pi}$ from week 6, which then gives us the ratio of a square to the correspond circle, which Wallis has shown to be the product on the left.