APPROXIMATION OF THE SUM OF THE INFINITE mod 4-ALTERNATING SERIES OF THE RECIPROCALS OF PRIMES BY EULER

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ABSTRACT: This paper presents a review of the alternating series of reciprocals of primes and walks through Euler's work in estimating and evaluating the series' value.

1. Introduction

In 1737, Euler showed that the infinite sum of reciprocals of all the primes,

$$\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\dots,$$

diverges and grows at the logarithm of the rate at which the harmonic series does. He remarks also that the sum of the reciprocals of all primes congruent to 1 (mod 4) and 3 (mod 4) diverge as well. Not only that, the sum of the reciprocals of all prime numbers satisfying a specific modular congruence diverges, provided that infinitely many such prime numbers satisfy said congruence [Eul].

Since all these series where the reciprocals have only positive signs diverge, Euler was inspired to investigate an alternating series of the reciprocals of primes [Eul07], where all primes congruent to $1 \pmod 4$ have a negative sign in front of their reciprocal and all primes congruent to $3 \pmod 4$ have positive sign:

$$\frac{1}{3}-\frac{1}{5}+\frac{1}{7}+\frac{1}{11}-\frac{1}{13}\dots$$

Even though this series does not, strictly speaking, alternate between positive and negative for each consecutive pair of terms, for the sake of simplicity we will refer to it throughout the paper as the alternating sum of reciprocals of primes. An important note is that this series hasn't been proved to converge or diverge, but it is suspected to converge.

In this paper, we present Euler's work, highlighting the thought process and reasoning, along with proofs of other theorems or lemmas he gives and uses without proof in his original paper.

2. Approximation with the summation using the Leibniz Series for π

To approximate the alternating sum, Euler starts out with a similar alternating series, the Gregory-Leibniz series for $\frac{\pi}{4}$:

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$$

Let this sum be represented by A.

Then, he eliminates the composite terms one by one as follows:

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Note that

$$\frac{1}{3}(A-1) = -\frac{1}{9} + \frac{1}{15} - \frac{1}{21} + \dots$$

$\frac{1}{3}(A-1)$ thus contains all the composite fractions with 3 as a divisor of the denominator, and all of these fractions now have the opposite sign as they did in A. This is because $3 \equiv -1 \pmod 4$, and all odd numbers are $\pm 1 \pmod 4$, so multiplying by $\frac{1}{3}$ negates the denominators' remainders while retaining the same sign in front of the fraction.

Thus, $A + \frac{1}{3}(A-1) = \frac{4A}{3} - \frac{1}{3}$ has no fractions whose denominator is a composite multiple of 3. Let $B = \frac{4A}{3} - \frac{1}{3}$.

The next step is to remove all the fractions with composite denominators that are multiples of 5.

$$B = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \dots,$$

so

$$B - 1 + \frac{1}{3} = \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \dots,$$

so $\left(B - 1 + \frac{1}{3}\right)$ contains the relevant multiples of 5 to remove. Now, since $5 \equiv 1 \pmod 4$, the signs of the denominators of $\left(B - 1 + \frac{1}{3}\right)$ are the same as when those denominators appear in B, so we subtract it from B instead.

Let the resulting series be $C = \frac{4B}{5} + \frac{1}{5}\left(1 - \frac{1}{3}\right)$.

Then, Euler keeps going, removing more and more fractions with composite denominators:

$$D = \frac{8}{7}C - \frac{1}{7}\left(1 - \frac{1}{3} + \frac{1}{5}\right)$$

$$E = \frac{12}{11}D - \frac{1}{11}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}\right)$$

$$F = \frac{12}{13}E + \frac{1}{13}\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11}\right)$$

Note that the leading fraction coefficient and the sign in front of the constants is slightly different depending on whether each fraction being factored out has a denominator congruent to $1 \pmod 4$ or $3 \pmod 4$.

We can keep listing out the sums in this fashion. Recall that the initial Gregory Leibniz series for π was represented by A. Using this, Euler plugs A into the successive equations for sums B, C, D, etc. Since the sums $B, C, D, E, F \dots$ approach the desired alternating sum of reciprocals of primes as more and more fractions with composite denominators are factored out, we can approximate the desired sum in this fashion.

When the sum is approximated in this way with all primes up to 29 removed, only the first two decimal places can be accurately determined, and the approximation is 0.331.

3. Approximation with the prime factorization of the Leibniz Series for π and $\zeta(2)$

To obtain another method for approximating the alternating sum of reciprocals of the primes, Euler factorizes the Gregory Leibniz series along with $\zeta_o(2)$, where $\zeta(s)$ is the Riemann Zeta function,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

and

$$\zeta_o(s) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s} = \left(1 - \frac{1}{2^s}\right)\zeta(s).$$

So,

$$\zeta_o(2) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{3}{4}\zeta(2) = \frac{\pi^2}{8}$$

3.1. **Factoring the Leibniz Series.** We will derive and prove the factorizations that Euler used. First, for any odd prime p, define r_p such that $r_p = \pm 1$ and $p \equiv r_p \pmod 4$.

Then, I claim that the Gregory Leibniz series is equal to

$$P = \prod_{p>2,p \text{ prime}}^{\infty} \sum_{i=0}^{\infty} \left(\frac{r_p}{p}\right)^i$$

To prove this, first note that all the prime factorizations of the odd positive integers will be generated in the expansion of P, as the product is taken over the sum of all powers of all odd primes.

All that remains to be shown is that the signs of the fractions alternate as in the Gregory Leibniz series.

In the Gregory Leibniz series, every fraction whose reciprocal is congruent to $1 \pmod 4$ has a positive sign in front of it while every fraction whose reciprocal is congruent to $3 \pmod 4$ has a negative sign in front of it.

If an odd positive integer is congruent to $1 \pmod 4$, then the total number of $3 \pmod 4$ primes dividing it, including repeats, must be even.

This is because any $1 \pmod 4$ primes have an r_p value of 1 and does not change the number's remainder $\pmod 4$, while multiplying by a $3 \pmod 4$ prime, with $r_p = -1$, negates the number's remainder $\pmod 4$.

So, any odd positive integer is congruent to $(-1)^k \pmod 4$, where k is the number of $3 \pmod 4$ primes dividing it, including repeats.

Meanwhile, when P is expanded, each fraction's sign is determined by the product of the r_p values for the primes dividing it. Once again, $1 \pmod 4$ primes have r_p values of 1, so they do not affect the sign.

However, each $3 \pmod 4$ prime term contributes a -1 if they are raised to an odd power or 1 if they are raised to an even power. This means that if a number has an odd total number of $3 \pmod 4$ primes dividing it, its reciprocal will have a negative sign and a positive sign for an even total number of $3 \pmod 4$ primes, just as before.

So, P and the Gregory Leibniz series are the same.

Then, since $r_p = \pm 1$, we can write

$$P = \frac{\pi}{4} = \left(\prod_{\substack{p \equiv 1 \pmod 4 \\ p \text{ prime}}} \frac{p}{p-1} \right) \left(\prod_{\substack{q \equiv 3 \pmod 4 \\ q \text{ prime}}} \frac{q}{q+1} \right) = \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{7}{8} \cdot \frac{11}{12} \dots$$

by the geometric series formula.

3.2. **Factorization of $\zeta_o(2)$.** Next, we want to factor

$$\zeta_o(2) = \sum_{i=0}^{\infty} \frac{1}{(2n+1)^2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$$

This is just equal to

$$\prod_{p \neq 2, p \text{ prime}} \sum_{i=0}^{\infty} \frac{1}{p^{2i}} = \prod_{p \neq 2, p \text{ prime}} \frac{p^2}{p^2-1} = \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{5 \cdot 5}{4 \cdot 6} \cdot \frac{7 \cdot 7}{6 \cdot 8} \dots = \frac{\pi^2}{8}$$

Then, if we square P and divide it by $\zeta_o(2)$, we get

$$\left(\prod_{\substack{p \equiv 1 \pmod 4 \\ p \text{ prime}}} \frac{p-1}{p+1} \right) \left(\prod_{\substack{q \equiv 3 \pmod 4 \\ q \text{ prime}}} \frac{q+1}{q-1} \right) = 2$$

Taking the logarithm and expanding, we have

$$\ln 2 = \ln \frac{3+1}{3-1} + \ln \frac{5-1}{5+1} + \ln \frac{7+1}{7-1} + \ldots$$

3.3. **Converting the logarithms into an infinite series to approximate the alternating sum of reciprocals of primes.** In doing this next step, Euler used a lemma relating the natural logarithm to an infinite sum, which we will prove below.

Lemma 3.1.

$$\frac{1}{2} \ln \frac{a+1}{a-1} = \sum_{i=1}^{\infty} \frac{1}{(2i-1)a^{2i-1}}$$

Proof. To prove this, start with the infinite geometric series

$$\frac{1}{a^2-1} = \frac{\frac{1}{a^2}}{1-\frac{1}{a^2}} = \sum_{i=1}^{\infty} \frac{1}{a^{2i}}$$

$$\frac{-2}{a^2-1} = \frac{\frac{-2}{a^2}}{1-\frac{1}{a^2}} = \sum_{i=1}^{\infty} \frac{-2}{a^{2i}}$$

$$\frac{1}{\frac{a+1}{a-1}} \cdot -\frac{2}{(a-1)^2} = -\frac{2}{(a+1)(a-1)} = \frac{-2}{a^2-1} = \frac{\frac{-2}{a^2}}{1-\frac{1}{a^2}} = \sum_{i=1}^{\infty} \frac{-2}{a^{2i}}$$

Note that $\frac{a+1}{a-1} = 1 + \frac{2}{a-1}$, so $\left(\frac{a+1}{a-1}\right)' = \left(\frac{2}{a-1}\right)' = -\frac{2}{(a-1)^2}$

Then, we can integrate both sides of the geometric series sum above using the chain rule on the left hand side:

$$\ln \frac{a+1}{a-1} = \left(\sum_{i=1}^{\infty} \frac{2}{(2i-1)a^{2i-1}}\right) + C,$$

where C is a constant of integration.
Taking the limit as $a \to \infty$ on both sides, we get $\ln 1 = 0 + C$, and thus $C = 0$.
So,

$$\ln \frac{a+1}{a-1} = \left(\sum_{i=1}^{\infty} \frac{2}{(2i-1)a^{2i-1}}\right),$$

and

$$\frac{1}{2} \ln \frac{a+1}{a-1} = \left(\sum_{i=1}^{\infty} \frac{1}{(2i-1)a^{2i-1}}\right)$$

as desired.

□

As a direct result of the above lemma, we also have

$$\frac{1}{2} \ln \frac{a-1}{a+1} = -\left(\sum_{i=1}^{\infty} \frac{1}{(2i-1)a^{2i-1}}\right)$$

To apply this lemma to the above sum, Euler first multiplies all terms by $\frac{1}{2}$:

$$\frac{1}{2} \ln 2 = \frac{1}{2} \ln \frac{3+1}{3-1} + \frac{1}{2} \ln \frac{5-1}{5+1} + \frac{1}{2} \ln \frac{7+1}{7-1} + \ldots$$

Then, he plugs in the corresponding infinite series for each prime:

$$\frac{1}{2} \ln \frac{3+1}{3-1} = \frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \ldots$$

$$\frac{1}{2} \ln \frac{5-1}{5+1} = -\frac{1}{5} - \frac{1}{3 \cdot 5^3} - \frac{1}{5 \cdot 5^5} - \ldots$$

After substituting all of the sums in, he defines

$$O = \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{11} - \frac{1}{13} - \ldots$$

$$P = \frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} - \frac{1}{13^3} - \ldots$$

$$Q = \frac{1}{3^5} - \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} - \frac{1}{13^5} - \ldots,$$

and so on.
Note that O is the same as the original alternating sum he is trying to evaluate and approximate.
Then, plugging these sums into the equation for $\ln 2$ gives

$$\frac{1}{2} \ln 2 = O + \frac{1}{3} P + \frac{1}{5} Q + \ldots$$

Solving for O, we have

$$O = \frac{1}{2}\ln 2 - \frac{1}{3}P - \frac{1}{5}Q - \dots$$

To approximate P, Q, etc., consider the sums of the form

$$1 - \frac{1}{3^n} + \frac{1}{5^n} - \frac{1}{7^n} - \dots,$$

where n is an odd positive integer.

Euler proceeds the same way as in the first method, subtracting and factoring out each odd prime number's composite multiples.

These sums are already known and given by Euler as

$$\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{(2i-1)^{2n+1}} = (-1)^n \frac{E_{2k}}{2(2k)!} \left(\frac{\pi}{2}\right)^{2n+1},$$

where E_n is the nth Euler number [EL04].

Thus, when Euler approximates P, Q, etc. with the infinite factoring and subtraction method, he obtains a slightly better result due to smaller errors from the higher powers.

This gives a much better approximation of 0.3349816, so the sum is greater than $\frac{1}{3}$.

4. Closed form and Patterns

Unfortunately, the sum of alternating reciprocals of odd primes does not have a known closed form, and no discernable pattern could be found connecting the sum to the alternating sums of higher powers.

5. Conclusion and Discussion

In conclusion, Euler was able to cleverly approximate the given series by noticing other series with similar terms, factorizations, and alternating signs and using them to aid his calculations. In addition to progress on the series, the work also gives great insight on how many similar infinite series were studied and evaluated. However, this series is still elusive due to the limited knowledge about primes and prime gaps, and convergence and closed forms are still unknown to this day.

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