EULERIAN NUMBERS

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1. Introduction

Eulerian numbers are combinatorial numbers, which were first introduced by Leonard Euler in his book *Institutiones calculi differentialis* [Eul]. Euler's original definition of Eulerian numbers is given in section 3. A modern reference for Eulerian numbers is Peterson's book *Eulerian numbers* [Pet]. We highly recommend the reader to read Peterson's book if the reader develops an interest in Eulerian numbers.

2. Eulerian Numbers

Definition 2.1 (Symmetric Group)**.** Given a natural number n, the *symmetric group* S_n, is the set of all permutations of $\{1, 2, 3, \ldots, n\}$(i.e bijections, $\sigma : \{1, 2, 3, \ldots, n\} \to \{1, 2, 3, \ldots, n\}$).

To avoid clutter it is useful to write a permutation in the so called *one line* notation, that is $\sigma = \sigma(1)\sigma(2)\ldots\sigma(n)$. So an example of an element in S_3 is $\sigma = 213$, i.e. the permutation that maps $1 \to 2$, $2 \to 1$ and $3 \to 3$.

Definition 2.2 (Descents)**.** Given a permutation, $\sigma \in S_n$ an index i is said to be a *descent*, if $\sigma(i) > \sigma(i + 1)$.

Remark 2.3. A function from $\bigcup S_n$ to $\mathbb{N}$ is called a *permutation statistic*, the function mapping a permutation to the number of descents it has is an example of a permutation statistic.

We are now ready to define Eulerian numbers.

Definition 2.4. An *Eulerian number*, $\left\langle {n \atop k} \right\rangle$, is the number of permutations, $\sigma \in S_n$ such that the number of descents σ has is equal to k.

Example. Let us compute the Eulerian number:

$$\left\langle {3 \atop 1} \right\rangle$$

The permutations in S_3 are:

$$123, 132, 213, 231, 312, 321$$

Of these the ones with 1 descent are:

$$132, 213, 231, 312$$

Therefore $\left\langle {3 \atop 1} \right\rangle = 4$.

The following is the table of the first few Eulerian numbers:

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$\left\langle {n \atop k} \right\rangle$ values for $n \leq 10$

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
0	1									
1	1									
2	1	1								
3	1	4	1							
4	1	11	11	1						
5	1	26	66	26	1					
6	1	57	302	302	57	1				
7	1	120	1191	2416	1191	120	1			
8	1	247	4293	15619	15619	4293	247	1		
9	1	502	14608	88234	156190	88234	14608	502	1	
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

Theorem 2.5. *The rows of the table are palindromic.*

Proof. Given a permutation in S_n, $n > 0$, with k descents we can get a one to one correspondence with a permutation in S_n with $n-1-k$ descents by reversing the permutation, this implies that $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle$ for $n > 0$, and therefore the rows of the permutation are palindromic. ■

Theorem 2.6. *The nth row of the table adds to $n!$, in other words $\sum_{k=0}^{n} \left\langle {n \atop k} \right\rangle = n!$.*

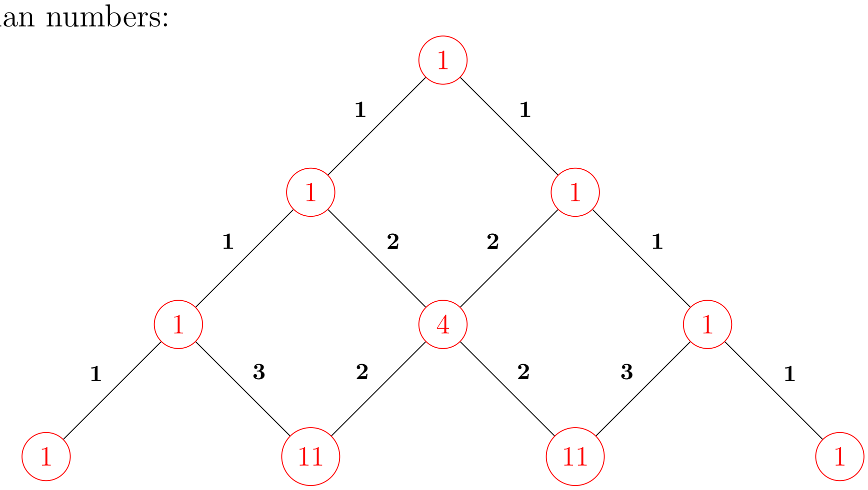
Proof. The sum of all numbers in the nth row is equal to the total number of permutations in S_n, since this number is $n!$, we get the desired equality. ■

One has the following reccurance-relation for the Eulerian numbers:

Theorem 2.7. *For any $n > 0$ and all k, $\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle$.*

Proof. Given a permutation in S_n with k descents, we can delete the number n from the one line notation to obtain a permutation in S_{n-1} with k or $k-1$ descents. Given a permutation in S_{n-1} with k descents we can add n in $k+1$ positions to maintain k descents. Given a permutation in S_{n-1} with $k-1$ descents we can add n to $n-k$ positions to get k descents. Adding these two values we get the desired result. ■

The recurrence relation gives us the following illuminating "weighted" pascal-like triangle for the Eulerian numbers:



One of the most interesting identities related to the Eulerian numbers is known as *Worpitzky's identity*, which we state below:

Theorem 2.8 (Worpitzky's identity)**.**

$$(k+1)^n = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+n-i}{n}$$

Proof. We shall give a bijective proof of the identity. Define a *barred permutation* to be a permutation in S_n with k inserted bars, with the restriction that at least one bar must be inserted between a descent. Let $B(n,k)$ denote the number of barred permutations in S_n with k bars. For example $B(3,2) = \text{Card}(\{||123, |3|12, 3|2|1, ...\})$. We can count $B(n,k)$ in two different ways, we can obtain a barred permutation of n with k bars from a normal permutation with i descents by placing a bar between each descent and then placing $k-i$ bars, the total number of ways to do this is $\left\langle {n \atop i} \right\rangle \binom{n+k-i}{n}$ therefore we have $B(n,k) = \sum_{i=0}^{n-1} \left\langle {n \atop i} \right\rangle \binom{k+n-i}{n}$, but we can count $B(n,k)$ in a different way, since the numbers between any two bars are increasing we have that $B(n,k) = (k+1)^n$, equating the two values we got, we get the desired equality. ∎

We can apply Worpitzky's identity repeatedly to get the following:

$$\left\langle {n \atop 0} \right\rangle = 1$$

$$\left\langle {n \atop 1} \right\rangle = 2^n - (n+1)$$

$$\left\langle {n \atop 2} \right\rangle = 3^n - 2^n(n+1) + \binom{n+1}{n-1}$$

Continuing in this manner we get the following alternating sum formula:

Theorem 2.9 (Alternating sum formula)**.** *For any $n \geq 1$ and all k,* $\left\langle {n \atop k} \right\rangle = \sum_{i=0}^{k} (-1)^i (k+1-i)^n \binom{n+1}{i}$

3. Eulerian Polynomials

(**Note**: We shall completely disregard convergence issues in this section.)

Remark 3.1. Historically it were the Eulerian polynomials that gave rise to the Eulerian numbers.

Notice that

$$\sum_{k=0}^{\infty} kx^k = \frac{x \cdot 1}{(1-x)^2}$$

Differentiating both sides we get:

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{x(1+1x)}{(1-x)^3}$$

Continuing this process of differentiating we get:

$$\sum_{k=0}^{\infty} k^3 x^k = \frac{x(1 + 4x + 1x^2)}{(1-x)^4}$$

$$\sum_{k=0}^{\infty} k^4 x^k = \frac{x(1 + 11x + 11x^2 + 1x^3)}{(1-x)^5}$$

As one can see the Eulerian numbers have emerged in the coefficients of the polynomials.

Definition 3.2. An *Eulerian Polynomial* is a polynomial, $A_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$. For convenience we define $A_0(x) = 1$(unfortunately this conflicts with the usual empty sum convention).

Using induction and the recurrence for the Eulerian numbers, we get the following recurrence for the Eulerian polynomials:

Theorem 3.3.

$$A_{n+1}(x) = (1 + nx)A_n(x) + x(1-x)A_n'(x)$$

Induction and the recurrence for the Eulerian polynomials allows us to prove the curious observation made at the beginning of the section, the identity is known as the *Carlitz Identity*.

Theorem 3.4 (Carlitz Identity)**.**

$$\sum_{k=0}^{\infty} k^n x^k = \frac{x \cdot A_n(x)}{(1-x)^{n+1}}$$

To finish this section let us present a cool application of the Carlitz identity. Recall from Simon's notes [RS, Chapter 5] we have:

$$\zeta(-k) = (-1)^k \frac{B_{k+1}}{k+1}$$

and

$$\zeta_a(-k) = (1 - 2^{k+1})\zeta(-k)$$

Where the B_k are the Bernoulli numbers and k is a non-negative integer.

We can also evaluate $\zeta_a(-k)$, using the Carlitz identity. Letting $x = -1$ in the Carlitz identity, we get that:

$$\zeta_a(-k) = \frac{-A(-1)}{2^{k+1}}$$

Solving for $A(-1)$ we get that:

$$A(-1) = 2^{n+1}(2^{n+1} - 1)\frac{B_{n+1}}{n+1}$$

In other words we have:

$$\boxed{\sum_{m=0}^{n} (-1)^m \left\langle {n \atop m} \right\rangle = (2^{n+1} - 1)\frac{B_{n+1}}{n+1}}$$

The boxed identity gives us a cool relation between the Eulerian numbers and the Bernoulli numbers.

4. Generating function for the Eulerian polynomials

Let us give the definition of an *exponential generating function*:

Definition 4.1. Given a sequence, a_n, the exponential generating function for the sequence a_n is the series $f(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}$.

Consider the exponential generating function for the sequence of Eulerian polynomials, $A(x,t) = \sum_{n=0}^{\infty} A_n(x) \frac{t^n}{n!}$, then we have that by plugging in for $A_n(x)$ using the Carlitz identity we have that $A(x,t) = \frac{x-1}{x-e^{t(x-1)}}$.

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