DIOPAHNTINE PROBLEMS ON THE QUARTIC FORMS

VAISHNAVI VENKAT

1. Abstract

This paper aims to examine Euler's techniques in solving quartic Diophantine equations and their lasting impact on mathematics.

2. Introduction

Diophantine equations are polynomial functions in two or more variables that are solved only for integer solutions. Such equations with integer solutions, which are mostly positive, are more sought after in many practical fields. This makes the process of deriving the solution much more crucial to various applications.

But the problem with the fourth-degree Diophantine equation was the complex interactions between the different variables, considering there was no particular way to generalize such equations to isolate variables.

Leonhard Euler, a prominent mathematician of the 18th century, made remarkable contributions to various mathematical fields, including number theory and Diophantine equations. Euler's exploration of quartic Diophantine equations, characterized by fourth-degree polynomials, exemplified his ingenuity.

Despite these challenges, Euler made significant contributions to the field of Diophantine equations. He developed innovative methods and solved numerous individual equations, but his work was mostly case-specific rather than providing a general theory.

Euler's efforts in solving these equations provided insights into integer solutions and influenced subsequent research.

3. Background and Historical Context

3.1. **Development of Quartic Diophantine Equations Through History.** Euler's contributions built upon earlier work, extending methods for solving lower-degree equations. Euler made significant contributions to solving Diophantine equations, particularly by building upon the work of Pierre de Fermat. He used a combination of modular arithmetic and infinite descent and introduced the Totient function and generalized Fermat's Little theorem to tackle lower-degree Diophantine equations.

Quartic Diophantine equations presented challenges due to variable interplay and the lack of a general solution approach.

Euler's interest in Diophantine equations stemmed from his broader number theory pursuits. His innovative strategies left a lasting impact on Diophantine equation research.

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4. Statement of the Problem

4.1. **Introduction to Euler's Quartic Diophantine Equation: $x^4 + y^4 = z^4$.** Euler explored the equation $a^2x^4 + 2abx^3y + cx^2y^2 + 2bdxy^3 + d^2y^4$, seeking integer solutions.

4.2. **Significance of the Quartic Diophantine Equation in Number Theory.** The quartic Diophantine equation contributes to understanding fourth powers and integer relationships.

5. Euler's Contributions

Euler applied modular arithmetic, infinite descent, and Euler's totient function.

Euler made significant contributions to solving Diophantine equations. Notably for lower degree equations:

5.1. **Euler's Totient Function.** The Euler's totient function, denoted as $\phi(n)$, counts positive integers less than n coprime to n:

$$\phi(n) = \left|\{m \in \mathbb{Z}^+ : 1 \leq m < n, \gcd(m, n) = 1\}\right|$$

For example, $\phi(8) = 4$, as there are four such integers for 8.

5.2. **Fermat's Little Theorem and Euler's Generalization.** Fermat's Little Theorem: If p is prime, $a^{p-1} \equiv 1 \pmod{p}$ for a coprime to p.

Euler's Generalization: For coprime a and n, $a^{\phi(n)} \equiv 1 \pmod{n}$.

Using these, Euler solved various lower-degree Diophantine equations. E.g., finding remainder of 5^{21} divided by 11:

$$5^{21} = (5^{10})^2 \times 5 \equiv 1^2 \times 5 \equiv 5 \pmod{11}.$$

5.3. **Fermat's Method of Infinite Descent.** Infinite descent is a technique to prove the absence of integer solutions by contradiction. This method relies on the intuition that a series cannot be infinite if it is of only non-negative integers and is strictly decreasing.

For example, consider the equation $x^4 + y^4 = z^2$ where $xyz \neq 0$

Here it is possible to assume that x^2, y^2, z are co-prime, since even if they are not, their common factors would cancel out and they would end up as co-primes.

Furthermore, the original equation could be written as $(x^2)^2 + (y^2)^2 = z^2$, which makes (x^2, y^2, z) a Pythagorean Triplet. There exist coprime p, p of opposite parity such that

$$x^2 = 2pq$$
$$y^2 = p^2 - q^2$$
$$z = p^2 + q^2$$

Now, these set of equations led to another Pythagorean Triplet with y, p, q, which further yields more similar equations with the sam restrictions,

$$q = 2ab$$
$$y = a^2 - b^2$$
$$p = a^2 + b^2$$

And since a, b have the same restrictions as p, q

$$x^2 = 2pq = 4ab(a^2 + b^2)$$

Now if $p \mid a$ or $p \mid b$, then it cannot divide $a^2 + b^2$ since a, b are co-prime. Which makes $a^2 + b^2$ and ab co-prime. Hence, ab and $a^2 + b^2$ are all perfect squares. Since ab is a perfect square, and a, b are relatively prime, both a and b are perfect squares, i.e. $a = A^2$, $b = B^2$. As $a^2 + b^2$ is a perfect square,

$$P^2 = a^2 + b^2 = A^4 + B^4$$

Since $P^2 = a^2 + b^2 = p < p^2 + q^2 = z$ and $P < Z$, it is evident that an infinite descent has occurred.

6. Euler's Quartic Diophantine Equation

Euler tackled fourth-degree Diophantine equations, like $A^4 + B^4 + C^4 = D^4$. using the method explained in his E772 paper mainly focused on the equation of the form

$$a^2x^4 + 2abx^3y + cx^2y^2 + 2bdxy^3 + d^2y^4 = \square$$

6.1. **Impact and Significance of Euler's Contributions.**
Euler's work laid the foundation for higher-degree Diophantine equation research. like: Matiyasevich's result emphasizes limitations of algorithmic solutions, Euler's conjecture was disproved, leading to exploration of solutions and parametric solutions and Mathematicians like Ramanujan and Jacobi contributed special solutions and identities.

7. Euler's Approach

Quartic equations posed the difficulty of isolating variables to find integer solutions since there isn't much information or general formulas to derive roots of biquadratic (quartic) problems. Euler dealt with equations of the form:

$$a^2x^4 + 2abx^3y + cx^2y^2 + 2bdxy^3 + d^2y^4 = \square$$

where he lets the equation equal an unknown value.

Euler used Number theory and Algebraic Manipulation to represent such problems in Quadratic form. Although Euler's method was not a general formula like for quadratic equations, the approach was applicable not only for special case scenarios but for many Diophantine quartic problems more generally.

On observation, Euler noticed that the previous equation reduces to a second-degree expression under quantity squared.

into the form

$$(ax^2 + bxy + dy^2)^2 + (c - b^2 - 2ad)x^2y^2$$

Consider the equation:

$$a^2x^4 + 2abx^3y + cx^2y^2 + 2bdxy^3 + d^2y^4$$

Grouping and rewriting the original quartic equation:

$$((ax^2)^2 + 2(ax^2)(bxy) + cx^2y^2 + 2(bxy)(dy^2) + d^2y^4).$$

With a few modifications the equation resembles the identity: $(a+b+c)^2 = a^2+b^2+c2+2ab+2bc+2ac$

Thus the equation can be written as

$$(ax^2)^2 + 2(ax^2)(bxy) + cx^2y^2 + 2(bxy)(dy^2) + (dy^2)^2 + 2(ax^2)(dy^2) - 2adx^2y^2 - b^2x^2y^2$$

Which simplifies to

$$(ax^2 + bxy + dy^2)^2 + (c - b^2 - 2ad)x^2y^2$$

Now, for the ease of working with simpler terms, Euler substituted the expressions within the brackets with more manageable variables i.e.

letting $(c - 2ad - b^2) = mn$

consequently, the other expression $(ax^2+bxy+dy^2)^2$ equates to $\lambda(mp^2-nq^2)$, for $xy = 2\lambda pq$ and on comparing the new equation with the original:

$$(ax^2)^2+2(ax^2)(bxy)+2(bxy)(dy^2)+(dy^2)^2+2(ax^2)(dy^2)+mnx^2y^2 = a^2x^4+2abx^3y+cx^2y^2+2bdxy^3+d^2y^4$$

.

Here m, n, p and q are considered integers, but if these variables take the value of a fraction or rational number, then y is considered one since the solution could only be an integer, which makes it binding for the denominator to equal one.

Now, the new value of x becomes $2\lambda pq$

substituting this value into the previous equation of

$$(ax^2 + bxy + dy^2)^2 = \lambda(mp^2 - nq^2)$$

yields

$$4a\lambda^2p^2q^2 + 2\lambda bpq + d - \lambda mp^2 - \lambda nq^2 = 0$$

Since it is a quadratic equation in two variables, p and q, the quadratic formula is used twice to represent both variables in terms of another

$$4a\lambda^2p^2q^2 + 2\lambda bpq + d - \lambda mp^2 - \lambda nq^2 = 0$$

$$(4a\lambda^2q^2 - \lambda m)p^2 + 2\lambda bpq + d - \lambda nq^2 = 0$$

By using the Quadratic formula:

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$p = \frac{-2\lambda bq \pm \sqrt{4\lambda^2b^2q^2 - 16a\lambda^2q^2d + 4a\lambda^3mq^2 - 4\lambda^2mnq^4}}{8a\lambda^2q^2 - 2\lambda m}$$

$$p = \frac{-\lambda bq \pm \sqrt{\lambda md + \lambda^2q^2(b^2 - 4ad + mn) - 4\lambda^3naq^4}}{4\lambda^2aq^2 - \lambda m}$$

Similarly, using the same steps for the variable q yields,

$$q = \frac{-2\lambda bp \pm \sqrt{4\lambda^2 b^2 p^2 - 16a\lambda^2 p^2 d + 4a\lambda^3 np^2 - 4\lambda^2 mnp^4}}{8a\lambda^2 p^2 - 2\lambda n}$$

$$q = \frac{-\lambda bp \pm \sqrt{\lambda nd + \lambda^2 p^2 (b^2 - 4ad + mn) - 4\lambda^3 map^4}}{4\lambda^2 ap^2 - \lambda n}$$

[Note: λ is mostly neglected from restrictions since it is simply an added variable that is left to our choice]

Due to the radical operation, the expression would produce two different values for p, which Euler referred to as p'.

$$q + q' = -\frac{2bp}{4\lambda app + n}$$

And since q is interdependent on p and vice versa, this process will yield a series of results for the values of p and q which are named as p', q', p'', q'' and so on, which follow the pattern;

$$p' = \left(-\frac{2bq}{4\lambda aq^2 - m}\right) - p; \quad q' = \left(-\frac{2bp'}{4\lambda ap'^2 + n}\right) - q$$

$$p'' = \left(-\frac{2bq'}{4\lambda aq'^2 - m}\right) - p''; \quad q'' = \left(-\frac{2bp''}{4\lambda ap''^2 + n}\right) - q'$$

$$p''' = \left(-\frac{2bq''}{4\lambda aq''^2 - m}\right) - p'''; \quad q''' = \left(-\frac{2bp'''}{4\lambda ap'''^2 + n}\right) - q''$$

And so on.

By rearranging the letters p and q:

We start with the original equations:

$$p' = \left(-\frac{2bq}{4\lambda aq^2 - m}\right) - p; \quad q' = \left(-\frac{2bp'}{4\lambda ap'^2 + n}\right) - q$$

.

On rearranging the first equation, which involves replacing the terms involving variables p and q with terms involving p' and q'. We'll use the original equation for q' to substitute for q. Starting with the original equation for p':

$$p' = \left(-\frac{2bq}{4\lambda aq^2 - m}\right) - p$$

.

Now, on substituting the original equation for q' into this equation:

$$p' = \left(-\frac{2b\left(-\frac{2bp'}{4\lambda ap'^2+n}\right)}{4\lambda a\left(-\frac{2bp'}{4\lambda ap'^2+n}\right)^2 - m}\right) - p$$

$$p' = \left(\frac{4b^2bp'}{(4\lambda a)^2 p'^2 - m(4\lambda ap'^2 + n)}\right) - p$$

$$p' = \left(\frac{4b^2bp'}{16\lambda^2 a^2 p'^2 - 4\lambda amp'^2 - mn}\right) - p$$

$$p' = \left(\frac{4b^2bp'}{p'^2(16\lambda^2a^2 - 4\lambda am) - mn} \right) - p$$

$$p' = \left(\frac{4b^2bp'}{p'^2(4\lambda a(4\lambda a - m)) - mn} \right) - p$$

Finally, rearrange the terms yields:

$$p' = \left(-\frac{2bq'}{4\lambda aq'^2 - m} \right) - p$$

.

Due to this rearrangement for both formulas of p and q series of equations for q, p, q', p', q'', p'' is obtained and defined as follows:

$$p' = \left(-\frac{2bq'}{4\lambda aq'^2 - m} \right) - p; \quad q' = \left(-\frac{2bp}{4\lambda ap^2 + n} \right) - q$$

$$p'' = \left(-\frac{2bq''}{4\lambda aq''^2 - m} \right) - p'; \quad q'' = \left(-\frac{2bp'}{4\lambda ap'^2 + n} \right) - q'$$

.

This modification to the formula for the values of p and q is important because it allows us to cut down on all the sets in between since just knowing the initial values of p and q creates a domino effect, producing the values of p', q', p'', q'' and so on.

This series allows us to generate multiple values for p and q without further substitution, leading to a progression that is more efficient than the usual method.

Furthermore, using the permuted formula, many more values of p and q are derived, and Euler noticed a law of progression that began to form.

Now with the infinite values of p and q, they can be combined in infinite different ways to get the value of x by using $\binom{N}{2}$, where N is the total number of values of p and q.

$$2\lambda pq, 2\lambda qp', 2\lambda p'q', 2\lambda q'p'', etc.$$

But, while the initial formula series produced another set of values for x, i.e.

$$2\lambda pq, 2\lambda q'p, 2\lambda p'q', 2\lambda q''p', etc.$$

These two sequences are just two possibilities of values for x. They are arranged in such a way that they follow a law of progression.

This was the general method that Euler used to obtain the values of x and y, the variables of the solution. And since initially xy was made equal to $2\lambda pq$, no matter what type of number $2\lambda pq$ ends up being, the values of x and y are manipulated in such a way that they always remain integers; Euler considered $y = 1$ if the value of $2\lambda pq$ turned out to be an integer, and in the other scenario where $2\lambda pq$ has a fractional value, then Euler allowed the value y to equal the denominator while x would take the value of the numerator. This would always work since recall that initially, $xy = 2\lambda pq$; thus, the fractional value of x would actually be $\frac{2\lambda pq}{y}$.

7.1. **Solving for p and q.** The initial values for p and q are the only values that need to be derived to solve for x and y. Euler names a common method to derive these initial values from the original fourth power equation.

He considers the scenario where $y = 1$, which makes the original equation a quartic expression in variable x

$$a^2x^4 + 2abx^3 + cx^2 + 2bdx + d^2$$

Now, Euler sets the root of the original fourth power expression to several values that resemble a variation of $\lambda(mp^2 - nq^2)$, which would be beneficial while trying to manipulate the formula for x to derive the value of p and q.

[Note: This was possible since Euler did not clearly define what the original fourth power expression would be equal to and thus used this ($\square$) to indicate its value. Therefore this means that in this case, a root need not equate the expression to zero]

First, Euler sets the root to be $ax^2 + bx - d$ or $\frac{c-b^2}{2d}x^2 + bx + d$, is similar to the previous equation but with certain rearrangements. On plugging into the original equation and solving for x offers the formula:

$$x = \frac{4bd}{b^2 - 2ad - c} = \frac{-4bd}{mn + 4ad}$$

Since, $c = mn + b^2 + 2ad$

Similarly setting the root to be equal to $ax^2 - bx - d$ or in a different form $ax^2 + bx + \frac{c-bb}{2a}$, which allows x to be:

$$x = \frac{b^2 - 2ad - c}{4ab} = \frac{-mn - 4ad}{4ab}$$

It is important to notice that although it may seem like this is the solution for x, this is not based on the original equation but on a modified version where a root is substituted into the equation.

Euler describes a process of finding suitable values for variables x, p, and q based on given equations. Let's break it down more concisely:

Starting with $y = 1$ to find a value for x.

$$ax^2 + bx + d = \lambda(mpp - nqq)$$

where $x = 2\lambda pq$ when $y = 1$. This results in

$$\frac{ax^2 + bx + d}{x} = \frac{mpp - nqq}{2pq}$$

Given $ax^2 + bx + d = Ax$, and $mpp - nqq = 2Apq$, it's deduced that

$$p = \frac{A + \sqrt{A^2 + mn}}{m}$$

, which can be simplified further.

By considering $p = f$ and $q = g$, a fraction f/g can be formed, and a value for λ is determined such that $2\lambda pq = x$. This leads to the formation of the mentioned series.

The method is exemplified through examples, building upon previous discussions in earlier sections.

8. Generalization of Euler's Method and Example

Euler, in one of his earlier papers (E763), focused on a special case of the process followed in his E772 paper.

Although this method is not quite general, Euler mentions a few examples in his paper which allow the same techniques to be applied to other quartic Diophantine equations by transforming them through algebraic manipulation to resemble the original equation used in his E772 paper.

One of the examples that follow this mentioned in his paper is based on a more general equation,

$$\alpha A^4 \pm \beta B^4 = \square$$

The formula involves variables α, β, A, B. It introduces a simpler version when $A/B = C$, and when specific conditions are met, the formula simplifies further. This is related to the substitution $C = \frac{1+x}{1-x}$ and its application.

The formula $\alpha A^4 \pm \beta B^4 = \square$ is presented, where α and β are coefficients, and A and B are variables raised to the fourth power.

When $A/B = C$, the formula simplifies to $\alpha C^4 \pm \beta = \square$. This version is easier to get the expression to resemble the initial equation.

In cases where $C = 1$ or $A = B$, the formula becomes $\alpha \pm \beta = \square$, resulting in a perfect square.

All formulas of this type can be transformed into a common form using the substitution $C = \frac{1+x}{1-x}$. By setting $\alpha + \beta = a^2$, the formula adopts the following structure:

$$a^2 + 4(\alpha - \beta)x + 6a^2x^2 + 4(\alpha - \beta)x^3 + a^2x^4 = \square$$

It is clear that in this equation, $a = d$, which would now look more like the original formula:

$$(a + 2(\alpha - \beta)x + ax^2)^2 + 16\alpha\beta x^2$$

This process illustrates how these formulas can be connected and reduced using a specific substitution so that the same process can be applied all over again to solve the equation.

9. Significant Other Contributions to Solving the Quartic Diophantine Equations

Euler focused on quartic equations that were equal to an undefined value. Following the work of this work of Euler's other mathematicians worked on Quartic Diophantine equations mainly of the form, $x^4 + y^4 = z^4$

9.1. **Joseph-Louis Lagrange.**

Lagrange's work involved utilizing elliptic curves to solve quartic diophantine equations. He approached equations like $x^4 + y^4 = z^4$ using methods involving rational points on elliptic curves. He parameterized such equations using the well-known elliptic curve equation in Weierstrass form:

$$E : y^2 = x^3 - ax - b.$$

Lagrange's approach aimed to find points (x, y) on such elliptic curves that satisfied the quartic diophantine equations, effectively finding rational solutions to those equations.

Elliptic curve methods can provide parametric solutions and rational points on curves, which is valuable for generating solutions to quartic diophantine equations. They also offer a systematic framework for studying the equation's properties, but at the same time, they may become complex for certain types of quartic equations. They are only efficient when dealing with equations that can be represented in elliptic curve form.

9.2. **Gabriel Lamé.** Lamé's work extended the theory of elliptic functions to quartic diophantine equations. For instance, consider the equation $x^4+y^4 = z^4$. Lamé's parametrization technique involved expressing x, y, and z in terms of Jacobi elliptic functions. These functions are solutions to the elliptic differential equation and have useful properties for studying diophantine equations.

Lamé's parametrization might look something like this:

$$x = \mathrm{sn}(u), \quad y = \mathrm{cn}(u), \quad z = \mathrm{dn}(u),$$

where $\mathrm{sn}(u)$, $\mathrm{cn}(u)$, and $\mathrm{dn}(u)$ are Jacobi elliptic functions with appropriate constants. Substituting these expressions into the quartic equation generates a parametric solution in terms of elliptic functions.

9.3. **Ernst Eduard Kummer.** Kummer's work with ideal numbers and regular primes laid the groundwork for understanding the solvability of quartic diophantine equations. Consider the equation $x^4 + y^4 = z^2$. Kummer showed that this equation has no nontrivial solutions in integers Kummer showed that this equation has no nontrivial solutions in integers x, y and z, when p is an odd prime and $p \equiv 1 \pmod{16}$. This result showcases his approach to using higher algebraic techniques, including ideal numbers, to study diophantine equations.

For example, when we consider $p = 17$, then using this method we would consider $(x^4+y^4 = z^2)$ modulo 17, i.e. evaluating the fourth powers modulo 17:

$$\begin{aligned}
0^4 &\equiv 0 \pmod{17}\\
1^4 &\equiv 1 \pmod{17}\\
2^4 &\equiv 16 \pmod{17}\\
3^4 &\equiv 13 \pmod{17}\\
4^4 &\equiv 1 \pmod{17}\\
5^4 &\equiv 16 \pmod{17}\\
6^4 &\equiv 1 \pmod{17}\\
7^4 &\equiv 16 \pmod{17}\\
8^4 &\equiv 1 \pmod{17}\\
9^4 &\equiv 0 \pmod{17}
\end{aligned}$$

and so on,

But since there are no values such that $k^4 \equiv 2$ or $k^4 \equiv 3 \pmod{17}$. Thus the equation $(x^4 + y^4 = z^2)$ has no trivial solutions or $p = 17$. This demonstrates Kummer's method of determining the solvability of a diophantine equation, by by analyzing the equation modulo a regular prime.

Kummer's method involved factoring the left-hand side of the equation modulo a regular prime p and analyzing the powers of p that divide the factors. If certain conditions were met, he concluded that no solutions existed.

Here as mentioned before, Euler's Totient Function, Fermant's Little Theorem, and Euler's generalization play a major role in solving quartic diophantine equations.

9.4. **Leopold Kronecker.** Kronecker's algebraic methods contributed to the study of quartic diophantine equations indirectly through his emphasis on algebraic number theory. His introduction of algebraic integers and ideals laid the foundation for modern approaches to diophantine equations using algebraic techniques.

Kronecker's ideas allowed mathematicians to relate the properties of algebraic number fields to diophantine equations, including quartic ones. By considering the properties of number fields and their extensions, mathematicians gained insights into the structure of solutions to diophantine equations involving quartic terms.

9.5. **Ferdinand von Lindemann.** Lindemann's proof of the transcendence of π relied on the Lindemann–Weierstrass theorem, which states that if $a_1, \ldots, a_n$ are algebraic numbers (numbers that are solutions of polynomial equations with integer coefficients), and $b_1, \ldots, b_n$ are nonzero algebraic numbers, then the number $a_1^{b_1} \cdots a_n^{b_n}$ is transcendental.

Lindemann's proof essentially demonstrated that if a is an algebraic nonzero number and b is an irrational algebraic number, then a^b is transcendental. This result has implications for diophantine equations involving exponential terms, such as those present in quartic equations.

For example, Lindemann proved that $e^x = 3$ has no algebraic solutions using transcendence proof.

In this case,

$a = e$ and $b = ln3$ where ln is the natural log. Since $ln3$ is irrational e is a nonzero algebraic number (as it is the base of the natural logarithm), Lindemann's theorem implies that $e^{ln3} = 3$ is transcendental, and therefore there are no algebraic solutions to x in $e^x = 3$

is transcendental. Therefore, there are no algebraic solutions for x in the equation

This result demonstrates Lindemann's approach of proving that certain equations involving algebraic and transcendental numbers have no algebraic solutions and in the process working on quartic Diophantine equations.

10. Conclusion

Euler's contributions were not merely definitive solutions, but guiding lights that paved the way for subsequent generations of mathematicians. While his work may not have provided the ultimate resolution to the complexity of quartic Diophantine equations, it served as a crucial foundation upon which future mathematicians could build more sophisticated methods.

Euler's ingenious techniques, marked by algebraic manipulation and creative parameterizations, not only deepened our understanding of the subject matter but also set the stage for the likes of Leopold Kronecker and Joseph-Louis Lagrange to introduce more refined strategies. Kronecker's insights into the structure of algebraic number fields and Lagrange's profound investigations into number theory were, in part, inspired by Euler's foundational work on quartic Diophantine equations, a part of mathematics that has become so widely used not only for other fields but also in the sciences.

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