EULER'S METHOD FOR FINDING LINEAR RELATIONS

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1. Linear Relations between 3 Numbers

A ratio between two numbers A and B is defined as $\alpha A = \beta B$, where α and β are the smallest integers possible. The same thing can be done for three numbers A, B, and C, where the relation would be in the form of

$$\alpha A = \pm\beta B \mp \gamma C.$$

We can find α, β, and γ in this relation by solving for them in the equation

$$\alpha A + \beta B + \gamma C = 0.$$

We do this by dividing the equation by the smallest of A, B, and C and setting the integral quotient and fractional remainder parts equal to a new variable. Then, we can use the fractional part to create an equation similar to our original one and keep repeating the process with these new equations until there is no fractional remainder, from where we can substitute backwards to find our original variables in terms of the last ones.

Example 1. To find a linear relation between the 3 co-prime numbers $A = 49$, $B = 59$, and $C = 75$, we can create the equation $49a + 59b + 75c = 0$. and use our method to solve for a, b, and c. First, we divide by the smallest number, which is 49, and separate the result of the division into the integral part, which we set equal to d, and the fractional part, which we set equal to -d. This gives us the equations

$$a + b + c = d$$

and

$$\frac{10b + 26c}{49} = -d$$

The terms in the second equation can be rearranged to give us a new equation, $10b + 26c + 49d = 0$, which is like our original equation. So, we can repeat the process, this time dividing by 10 and introducing the variable e to get

$$b + 2c + 4d = e$$

and

$$\frac{6c + 9d}{10} = -e$$

However, 6 and 9 share a factor of 3, so we can set the equations equal to 3e and -3e instead and then divide by three after rearranging, which leads to the equation $2c + 3d + 10e = 0$. Now, we repeat again, dividing by 2 and setting the results equal to f and -f to get

$$c + d + 5e = f$$

and

$$\frac{d}{2} = -f$$

This gives us $d + 2f = 0$, which does not create a fractional part when divided. So, we can stop here and work backwards to find the original variables. We start by substituting $d = -2f$ in the equation $c + d + 5e = f$ to get $c - 2f + 5e = f$, or $c = 3f - 5e$. Continuing to substitute in the same way, we get

$$b = 13e + 2f,$$

which gives us

$$a = -8e - 7f.$$

If we substitute the values for a, b, and c into the original equation, we get

$$-(8e + 7f)49 + (13e + 2f)59 + (3f - 5e)75 = 0,$$

which holds for any e and f. The simplest form of this uses $e = 0$ and $f = 1$, which gives us the final relation

$$7 \cdot 49 - 2 \cdot 59 - 3 \cdot 75 = 0.$$

2. Approximations for Irrational Numbers

There is no exact linear relation between irrational numbers, but it can be closely approximated using rounded decimal expansions.

Example 2. We can solve for the relation $a + b\sqrt{2} + c\sqrt{3} = 0$ by using the decimal expansions $\sqrt{2} \approx 1.414214$ and $\sqrt{3} \approx 1.732051$ and multiplying by 1000000 to get

$$1000000a + 1414214b + 1732051c = 0$$

By applying our method and dividing by 1000000, we get the equations

$$a + b + c = d$$

and

$$\frac{414214b + 732051c}{1000000} = -d,$$

which gives us $414214b + 732051c + 1000000d = 0$. Dividing this by 414214 gives us

$$b + c + 2d = e$$

and

$$\frac{317837c + 171572d}{414214} = -e.$$

We can keep repeating our process until we get to

$$49i + j + 56k = 0,$$

which will have no fractional part when it is divided by the smallest factor, 1. So, we can solve for j to get $j = -49i - 56k$ and substitute it into the previous equation $56g + 105i + 113j = 0$ to get

$$g = \frac{-105i - 113(-49i - 56k)}{56} = 97i + 113k,$$

which can be substituted into $169g + 113h + 557i = 0$ to give us

$$h = \frac{-557i - 169(97i + 113k}{113} = -150i - 169k.$$

We can keep repeating these substitutions backwards until we get to

$$c = -8313i - 9411k,$$

$$b = 63878i + 72424k,$$

$$a = -75938i - 86122k,$$

which gives us the approximate relation

$$(-75938i - 86122k) + (63878i + 72424k)\sqrt{2} + (-8313i - 9411k)\sqrt{3} \approx 0.$$

The simplest form for this uses $i = -1$ and $k = 1$, which leads to

$$-10184 + 8546\sqrt{2} - 1098\sqrt{3} \approx 0.$$

Computing the left side gives us approximately 0.0773173198, which means that the relation is close but not exact. This can also be demonstrated by squaring the unsolved relation $a = b\sqrt{2} + c\sqrt{3}$, which would give us $a^2 = 2b^2 + 3c^2 + 2bc\sqrt{6}$. Solving for $\sqrt{6}$ gives us $\frac{a^2-2b^2-3c^2}{2bc}$. Since a, b, and c are integers, the numerator and denominator are both integers, so the result is rational, even though $\sqrt{6}$ should be irrational. A similar thing can be done to any relation between irrational numbers that are not Galois conjugates, showing that there can be no exact linear relation between these numbers.

3. Relations with Transcendental Numbers

Transcendental numbers do not seem to be comparable with algebraic numbers. This has already been shown by the fact that the continued fraction for π does not seem to have any periodic terms, but we can see what happens when we try to find a relation.

Example 3. Let us find a relation with $a\sqrt{2} + b\sqrt{3} + c\pi = 0$. Like the previous example, we can use the decimal expansions to turn this into

$$1414214a + 1732051b + 3141593c = 0.$$

Dividing this by 141421 and applying our method gives us

$$a + b + 2c = d$$

and

$$\frac{317837b + 313165c}{1414214} = -d,$$

which can be turned into $317837b + 313165c + 1414214d = 0$. After dividing this by 313165 we get

$$c + b + 4d = e$$

and

$$\frac{4672b + 161554d}{313165} = e,$$

which can be reduced to $4672b + 161554d + 313165e = 0$. We now divide by 4672 to get

$$b + 34d + 67e = f$$

and

$$\frac{2706d + 141e}{4672} = f.$$

This is clearly getting more and more complicated, so even if there were an exact relation, it would be too complex to be useful. However, we know that there can be no exact relation between transcendental and algebraic numbers, as this is the basis of how transcendental numbers are defined.

We could also use our method to check if other possible types of transcendental numbers are related to circles or logarithms when we are not able to reduce them analytically.

Example 4. We can try to find a relation for the sum of the reciprocals of cubes

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \cdots ,$$

or $\zeta(3)$, which Euler was not able to express in terms of π or logarithms. It is now known as Apéry's constant, named after the French mathematician who proved that it is irrational. However, we still do not know if the constant is transcendental.

To find the relation, we notice that the sum seems like it should contain the cube of the alternating sum of first powers,

$$\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}+\cdots,$$

which we know converges to $\ln 2$. It should also include the product of this with the sum of reciprocals of squares,

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\cdots,$$

or $\zeta(2)$, which Euler had found to be $\frac{\pi^2}{6}$ in his solution to the Basel problem. This product is $(\ln 2\frac{\pi^2}{6})$, so we can form a relation using $\alpha(\ln 2)\frac{\pi^2}{6}$ and $\beta(\ln 2)^3$. However, we first need a shorter way to express the sum of reciprocals of cubes in the relation equation. Euler had previously approximated the sum of reciprocals of cubes to be 1.202056903, and we can get the alternating sum by subtracting one-fourths of this. This gives us

$$\frac{1}{1^3}+\frac{1}{2^3}-\frac{1}{3^3}+\frac{1}{4^3}-\frac{1}{5^3}+\cdots=0.901542677,$$

which we represent as A. Now, we can create the relation equation

$$aA+b(\ln 2)^3+c(\ln 2)\frac{\pi^2}{6}=0.$$

Again, we use approximations for the constants, using

$$\frac{\pi^2}{6}=1.644923066$$

and

$$(\ln 2)=0.693147180$$

to get

$$(\ln 2)^3=0.333025$$

and

$$(\ln 2)\frac{\pi^2}{6}=1.140182.$$

Substituting these into the equation and multiplying by 1000000 like in the previous examples gives us $901543a+333025b+1140182c=0$. We can now divide by 333025 and use our method to get

$$b+2a+3c=d$$

and

$$\frac{235493a+141107c}{333025}=d,$$

which can be simplified to $235493a + 141107c + 333025d = 0$. We divide by 14117 and repeat to get the equations

$$a + c + 2d = e$$

and

$$\frac{94386a + 50811d}{141107} = e.$$

Once again, the equations are just getting more and more complicated, and it seems impossible for them to resolve into an exact relation. So, our method for finding relations does not help us conclude whether or not this possibly transcendental sum can be expressed in terms of other transcendental numbers.