THE COMPLEX EXPONENTIAL FUNCTION

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1. Introduction

Leonhard Euler's complex exponential function established a connection between the exponential function and the trigonometric functions. To begin, the exponential function is defined as:

$$y = e^w x$$

In this case, we consider w as a constant, where if 0 was substituted for x, y would equal 1. On the other hand, imaginary numbers are negative numbers expressed in terms of a square root. This is denoted by:

$$i = (-1)^{\frac{1}{2}}$$

Furthermore, complex numbers are expressed in the form: $z = a + bi$, where a is the real part and b is the imaginary part. Considering a point on the unit circle where the x axis are the cosine values and the y axis are the sine values, consider Figure 1 as a representation of a point of the unit circle.

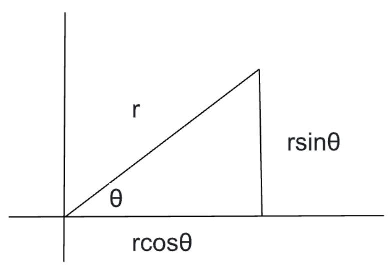

Figure 1. An image of the complex plane

Connecting Figure 1 to the complex plane, the x axis would represent the real part (a) of the complex number z while the y axis would represent the imaginary part (b) and r is a constant. Hence, in relation to the trigonometric functions, the real part of a complex number can be associated with cosine while the imaginary part can be associated with sine. Hence, another expression for complex numbers was established:

$$z = \cos x + i \sin x$$

Euler then contributed to this identity by deriving the complex exponential function:

$$e^{ix} = cosx + isinx$$

Therefore, this paper will discuss some of Euler's findings and also provide an extension of power series and polar coordinates proofs to derive this formula. It will also aim to evaluate its applications in hyperbolic functions and logarithmic functions.

Date: August 2023.

2. Power Series For Derivation

The power series is classified as a infinite series which is commonly used to detect the convergence of a series. In the case of Euler's complex exponential function, the power series of an exponential function can be used to derive the formula. Consider the power series for the exponential function e to be:

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots$$

In order to make the exponential complex, we can substitute t for ix, where x is an angle. In doing so we get:

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

Once this expression is simplified, we end up with:

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \cdots$$

Using the property $i = -1^{\frac{1}{2}}$ from the introduction, we can the simplify this expression to obtain:

$$e^{ix} = 1 + (ix) + \frac{-x^2}{2!} + \frac{-ix^3}{3!} + \frac{x^4}{4!} + \cdots$$

The next step of this proof is to consider the power series for the trigonometric functions $\sin x$ and $\cos x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Observe that the power series for $\sin x$ consists of odd coefficients and factorials of odd numbers, while the power series for $\cos x$ consists of positive coefficients and factorials of even numbers. If we consider the power series of the complex exponential once again, the pattern consists of both odd and negative coefficients for x with alternating signs. Hence, we can simplify it in terms of the power series of $\sin x$ and $\cos x$:

$$e^{ix} = (1 - \frac{-x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}) + ...i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...)$$

Finally we can state that:

$$e^{ix} = cosx + i(sinx)$$

We have proved Euler's complex exponential function using the power series. An advantage that the power series has over other proofs is that it consists of sequences that can be rearranged to find patterns of other power series, hence making it easier to simplify to the complex exponential function. Another method that can be used to prove this identity is using Polar Coordinates.

3. Polar Coordinates For Derivation

The polar coordinate system refers to when a point on a two-dimensional system is determined using a reference point and an angle taken from a reference point (Byjus). Referring to Figure 1 in the introduction, instead of referring to the point as (x,y), we can refer to the point as (r, θ). We now need to consider Euler's complex exponential identity:

$$e^{ix} = r(\cos\theta + i\sin\theta)$$

Assume that r is the radius, and θ is the angle from the reference point and that they both are a function of x. We can assume that our initial conditions of $r(0) = 1$ and $\theta(0) = 0$.If we differentiate for both sides, we get:

$$ie^{ix} = (\cos\theta + \sin\theta)\frac{dr}{dx} + r(-\sin\theta + i\cos\theta)\frac{d\theta}{dx}$$

In order to replace e^{ix}, we can substitute r(cosθ + isinθ) back in for it, obtaining the expression:

$$ir(\cos\theta + i\sin\theta) = \frac{dr}{dx}(cos\theta + sin\theta) + r(-sin\theta + i\cos\theta)\frac{d\theta}{dx}$$

By simplifying with i we obtain:

$$r(i\cos\theta - \sin\theta) = (\cos\theta + \sin\theta)\frac{dr}{dx} + r(-\sin\theta + i\cos\theta)\frac{d\theta}{dx}$$

Through this simplification, we have now obtained real and imaginary parts for each side of the equation. By now equating them according to their parts, we will create two separate equations:

$$ir\cos\theta = i\sin\theta\frac{dr}{dx} + ricos\theta\frac{d\theta}{dx}$$

$$-rsin\theta = cost\theta\frac{dr}{dx} - rsin\theta\frac{d\theta}{dx}$$

The next step involve replacing $\frac{dr}{dx}$ and $\frac{dx}{d\theta}$ with the variables a and b respectively to make the system of equations simpler to solve. We must then multiply the first equation by cosθ, and the second equation by sinθ, resulting in:

$$ircos^2\theta = icos\theta sin\theta a + ricos^2\theta b$$

$$-rsin^2\theta = cos\theta sin\theta a - rsin^2\theta b$$

If we multiply the second equation by -1 and solve for the system of equations. Hence, it results in:

$$r(cos^2\theta + sin^2\theta) = r(cos^2\theta + sin^2\theta)b$$

By solving the system of equations, we got rid of the variable 'a' and can now use the identity $cos^2\theta + sin^2\theta = 1$ to simplify it to

$$r = rb$$

which means that $b = 1$. Hence substituting for $b = 1$ in the original equation we obtain:

$$0 = (sin\theta)a$$

$$0 = (cos\theta)a$$

which indicates that a=0. Therefore, since a=$\frac{dr}{dx}$=0, we can assume r is a constant, wile on the other hand b=$\frac{dx}{d\theta}$ = 1 and therefore θ= x+C for a constant C. Our initial conditions stated that $r(0) = 1$, $\theta(0) = 0$ and hence the constant $C = 0$ and $x = \theta$. Therefore, we can conclude with the derivation stating:

$$e^{ix} = r(cos\theta + isin\theta)$$
$$= e^{ix} = cosx + isinx$$

4. Application in Hyperbolic Functions

Hyperbolic functions can be described as functions defined by the rotations along a hyperbola (Byjus).They are similar to trigonometric functions in the way that they're periodic and they're defined using the exponential function e^x and its inverse e^{-x}. Using Euler's complex exponential function, we can find the complex hyperbolic functions for $\sin x$, $\cos x$ and $\tan x$. Consider the identity:

$$= e^{ix} = cosx + isinx$$

We can then find it's inverse by substituting $-x$:

$$= e^{-ix} = cos(-x) + isin(-x)$$

We must now recall the theory of odd and even functions. Since $\cos x$ is an even function, which means it's symmetrical upon the y axis, then we can say $\cos(-x) = \cos x$. Since $\sin x$ is an odd function because divided along the y axis, the graph in the negative x axis is a reflection across the x axis of the graph in the positive x axis. Hence $\sin(-x) = -\sin x$. Therefore we can rewrite the inverse for the complex exponential function as:

$$= e^{-ix} = cos(x) - isin(x)$$

In order to obtain an expression for the hyperbolic function for complex $\cos x$, we can add the identities for the complex exponential function and its inverse:

$$= e^{ix} + e^{-ix} = \cos(x) + \cos(x) + i\sin(x) - i\sin(x)$$

Rearranging the equation of $\cos x$, and substituting $i\theta$ for x, we obtain:

$$2\cos(i\theta) = e^{i(i\theta)} + e^{-i(i\theta)}$$
$$\cos i\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$
$$= \cosh\theta$$

We can also find the hyperbolic equation for complex $\sin x$ by subtracting the two equations:

$$= e^{ix} - e^{-ix} = \cos(x) - \cos(x) + i\sin(x) + i\sin(x)$$

Rearranging for $\sin x$ and substituting $i\theta$ for x, we get:

$$= e^{i(i\theta)} - e^{-i(i\theta)} = 2i\sin(\theta)$$
$$\sin(i\theta) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

We can now multiply the numerator and denominator by i to get rid of i in the denominator:

$$\sin\theta = i(\frac{e^{-i\theta} - e^{i\theta}}{2})$$
$$= i\sinh\theta$$

To find the hyperbolic function for complex $\tan x$, we can divide the hyperbolic function for complex $\sin x$ by complex $\cos x$ to obtain:

$$\tan(i\theta) = \frac{\sin i\theta}{\cos i\theta}$$

$$= \frac{i \sinh \theta}{\cosh \theta}$$

$$= i \tanh \theta$$

5. Application in Complex Logarithms

In Euler's paper 858, Euler uses his complex exponential to derive a formula for complex logarithms. Consider the form:

$$e^{ix} = \cos x + i \sin x$$

He then apply logarithm of this equation on either side to obtain:

$$ix = log(\cos x + i \sin x)$$

. Using this formula he was able to find the logarithms of 1,-1, and i. Another formula that can be used for complex logarithms can be found by considering the form $z = a + bi$ where $'a'$ is the real part and $'b'$ is the complex part. Using a logarithmic function and its inverse(the exponential function) we can form a logarithm for z:

$$z = e^{lnz}$$

For any non-zero complex number, the exponential can be expressed as:

$$z = |z|e^{i\theta}$$

Using the first identity, we can state that:

$$z = |z|e^{i\theta} = e^{ln|z|}e^{i\theta}$$

$$= e^{ln|z|} + i\theta$$

Applying ln on either side of the equation, it can then simplify to

$$lnz = ln|z| + i\theta$$

To make this equation a general solution, we can consider θ to be of the form $\theta + 2k\pi$ where k belongs to all real integers. This helps us derive the formula:

$$lnz = ln|z| + i(\theta + 2k\pi)$$

where θ is between $-\pi$ and π. Therefore the complex exponential in this case helped derive the complex logarithm and has found a solution to the problem of periodicity of angles(mathvault).

6. Bibliography


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