ON A SERIES FORMED FROM RECIPROCALS OF PRIMES

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1. Approximating the Series Using the Leibniz Series

The topic of this paper is the series

$$\frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{11} - \frac{1}{13} - \frac{1}{17} + \cdots$$

Where every term is the reciprocal of a prime and each term of the form $\frac{1}{4n+1}$ has a negative coefficient, while each term of the form $\frac{1}{4n-1}$ has a positive coefficient. This series converges extraordinarily slowly, summing the first 25 terms of the series is only accurate to the tenths place. So our first goal will be to find a good approximation of the series.

Euler began by looking at the Leibniz series:

$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

He noticed that the denominators of this series contained every prime of the form $4n - 1$ with a minus sign, and those of the form $4n + 1$ with a plus sign. Therefore we can obtain the series we want complement to one by removing each composite term from the Leibniz series. To start, let's isolate the terms of the Leibniz series divisible by 3;

$A = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

$(A-1)\frac{1}{3} = -\frac{1}{9} + \frac{1}{15} - \frac{1}{21} + \cdots$

The above series contains the negative of every term of A divisible by 3. Adding to A, we get

$A + (A-1)\frac{1}{3} = 1 + \frac{1}{5} - \frac{1}{7} + \frac{1}{11} - \cdots = B$

Now we can continue this, removing every composite term until we are eventually left with the complement to one of the series we want to evaluate.

$(B - 1 + \frac{1}{3})\frac{1}{5} = \frac{1}{25} - \frac{1}{35} - \frac{1}{55} + \frac{1}{65} + \frac{1}{85} - \frac{1}{95} - \cdots$

$B - (B - 1 + \frac{1}{3})\frac{1}{5} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} + \cdots = C$

$C + (C - 1 + \frac{1}{3} - \frac{1}{5})\frac{1}{7} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \frac{1}{11} + \cdots = D\ \cdot$

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From these series, we can find that the sequence $1-A, 1-B, 1-C, 1-D, \cdots$ will approach the series we want to evaluate. Since we know that $A = \frac{\pi}{4}$, the subsequent series can be summed with relative ease, giving the approximate value of our series to be 0.331.

However we can get a far better approximation than this. Let's start by again removing every composite term from the Leibniz series, however this time we will remove the prime terms as well. This gives:

$A(1+\frac{1}{3}) = 1 + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \cdots = B$
$B(1-\frac{1}{5}) = 1 - \frac{1}{7} - \frac{1}{11} + \frac{1}{13} \cdots = C$
$C(1+\frac{1}{7}) = 1 - \frac{1}{11} + \frac{1}{13} + \frac{1}{17} \cdots = D$
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$A(1+\frac{1}{3})(1-\frac{1}{5})(1+\frac{1}{7})(1+\frac{1}{11}) \cdots = 1$
Multiplying both sides by the product of all primes, we get:

$A(3+1)(5-1)(7+1)(11+1) \cdots = 3*5*7*11*\cdots$
$A = \frac{\pi}{4} = \frac{3}{4}\frac{5}{4}\frac{7}{8}\frac{11}{12}\cdots$
Or $A^2 = \frac{\pi^2}{16} = \frac{3^2}{4^2}\frac{5^2}{4^2}\frac{7^2}{8^2}\frac{11^2}{12^2}\cdots$

With every prime number besides 2 in the numerator, and each multiple of 4 in the denominator such that each denominator is one more or less than the numerator. By again removing every term besides one from the series $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \cdots$ we find that $\frac{\pi^2}{8} = \frac{3^2}{2*4}\frac{5^2}{4*6}\frac{7^2}{6*8}\frac{11^2}{10*12}\cdots$. Than dividing by A^2, we find that $2 = \frac{3+1}{3-1}\frac{5-1}{5+1}\frac{7+1}{7-1}\frac{11+1}{11-1}\cdots$. Next, let's take the natural logarithm of both sides, giving:
$ln(2) = ln(\frac{3+1}{3-1}) + ln(\frac{5-1}{5+1}) + ln(\frac{7+1}{7-1}) + ln(\frac{11+1}{11-1}) + \cdots$
Which using the Taylor expansion of ln(1 - x), is equivalent to

$O = \frac{1}{2}ln(2) - \frac{1}{3}P - \frac{1}{5}Q - \frac{1}{7}R - \cdots$ Where O is the series $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{11} - \frac{1}{13} - \frac{1}{17} + \cdots$, and
$P = \frac{1}{3^3} - \frac{1}{5^3} + \frac{1}{7^3} + \frac{1}{11^3} - \frac{1}{13^3} - \frac{1}{17^3} + \cdots$
$Q = \frac{1}{3^5} - \frac{1}{5^5} + \frac{1}{7^5} + \frac{1}{11^5} - \frac{1}{13^5} - \frac{1}{17^5} + \cdots$
$R = \frac{1}{3^7} - \frac{1}{5^7} + \frac{1}{7^7} + \frac{1}{11^7} - \frac{1}{13^7} - \frac{1}{17^7} + \cdots$
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.
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etc

So now are goal will be to evaluate these series. To start, let's define these series:
$B(3) = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} - \frac{1}{11^3} + \frac{1}{13^3} + \frac{1}{17^3} - \cdots$
$B(5) = 1 - \frac{1}{3^5} + \frac{1}{5^5} - \frac{1}{7^5} - \frac{1}{11^5} + \frac{1}{13^5} + \frac{1}{17^5} - \cdots$
$B(7) = 1 - \frac{1}{3^7} + \frac{1}{5^7} - \frac{1}{7^7} - \frac{1}{11^7} + \frac{1}{13^7} + \frac{1}{17^7} - \cdots$
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In general,

$$B(s) = 1 - \frac{1}{3^s} + \frac{1}{5^s} - \frac{1}{7^s} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}$$

This is the Dirichlet B function. It is known that $B(3) = \frac{1}{2!} * \frac{\pi^3}{2^4}$, $B(5) = \frac{5}{4!} * \frac{\pi^5}{2^6}$, $B(7) = \frac{61}{6!} * \frac{\pi^7}{2^8}$, $B(9) = \frac{1385}{8!} * \frac{\pi^9}{2^{10}}$, Etc. In his original paper, Euler did not give a proof for these values, however I believe it would be beneficial to provide one here. To start, let's look at an infinite series expression for cotangent, namely:

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$$\pi cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{x+n} + \frac{1}{x-n} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2 - n^2}$$

Proof.

$$g(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \left(\frac{1}{x+n} + \frac{1}{x-n} \right)$$

$$g_N(x) = \frac{1}{x} + \sum_{n=1}^{N} \left(\frac{1}{x+n} + \frac{1}{x-n} \right) = \sum_{n=-N}^{N} \frac{1}{x+n}$$

Let's prove that $g(x)$ and $\pi cot(\pi x)$ share many properties. firstly, that both are defined for only real non-integer values of x. We can show that $\pi cot(\pi x)$ is only undefined on the integers through simple algebra, the tricky part is to show this is true for $g(x)$. First, if x is an integer in $\frac{1}{x} + \sum_{n=1}^{\infty} \frac{1}{x+n} + \frac{1}{x-n}$, than n must take on either x or minus x as n cycles from 1 to ∞. So their must be some value in the sum which is undefined, making the entire sum undefined. To show that $g(x)$ converges for non integer values of x, we can compare $g(x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2-n^2}$ to the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges. Thus, g(x) is defined if and only if $x \notin \mathbb{Z}$ and $x \in \mathbb{R}$

Next, let's show that both functions are periodic with period one. This is true for $\pi cot(\pi x)$ since both $\pi cos(\pi x)$ and $sin(\pi x)$ have period one. For $g(x)$, we have $g(x) - \lim_{N\to\infty} g_N(x)$, and

$$g_N(x+1) = \sum_{n=-N}^{N} \frac{1}{x+1+n} = \sum_{n=-N+1}^{N+1} \frac{1}{x+n} = g_N(x) + \frac{1}{x+N+1} - \frac{1}{x-N+1}$$

So we can split up the limit $\lim_{N\to\infty} g_N(x+1)$ into $\lim_{N\to\infty} g_N(x) + \lim_{N\to\infty} \frac{1}{x+N+1} - \frac{1}{x-N+1}$, given both limits exist. Since the ladder limit goes to zero for large N, we get $\lim_{N\to\infty} g_N(x+1) = \lim_{N\to\infty} g_N(x)$. or $g(x) = g(x+1)$, thereby showing that $g(x)$ has period 1.

Next, we aim to prove that both $g(x)$ and $\pi cot(\pi x)$ are odd functions. For $\pi cot(\pi x)$, since $\pi cos(\pi x)$ is even, and $sin(\pi x)$ is odd, then $\pi cot(\pi x)$ is odd. For $g(x)$, we can simply compute

$g(-x)$, giving:

$g(-x) = \lim_{N\to\infty} g_N(-x) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{-x+n} = \lim_{N\to\infty} \sum_{n=N}^{-N} \frac{1}{-x-n} = -g(x)$.

Next, let's show that both $g(x)$ and $cos(x)$ share the same functional equation, namely;

$f(\frac{1}{x}) + f(\frac{x+1}{2}) = 2f(x)$

For cotangent, this can be easily verified to become the sine and cosine addition laws, so we will only show that $g(x)$ satisfies the equation.

$$g_N(\frac{x}{2}) + g_N(\frac{x+1}{2}) = \sum_{n=-N}^{N} \frac{1}{x/2+1+n} + \sum_{n=-N}^{N} \frac{1}{(x+1)/2+1+n}$$

$= 2\sum_{n=-N}^{N} \frac{1}{x+1+n} + 2\sum_{n=-N}^{N} \frac{1}{x+1+n} = 2\sum_{n=-2N}^{2N+1} \frac{1}{x+1+n} = 2g_{2N}(x) + \frac{2}{x+2N+1}$

$g(\frac{x}{2}) + g(\frac{x+1}{2}) = \lim_{N\to\infty} 2g_{2N}(x) + \frac{2}{x+2N+1} = \lim_{N\to\infty} 2g_{2N}(x) = 2g(x)$
Which proves the functional equation for $g(X)$.

To show that $g(x) = \pi cot(\pi x)$, let's show that the function $h(x) = \pi cot(\pi x) - g(x)$ is identically zero. From the proven aspects of $\pi cot(\pi x)$ and $g(x)$, we know that $h(x)$ is continuous on $\mathbb{R}\backslash\mathbb{Z}$, has periodicity 1, is odd, and follows the functional equation shared by both $\pi cot(\pi x)$ and $g(x)$. Let's start with
$\lim_{x\to 0} h(x) = \lim_{x\to 0} \pi cot(\pi x) - \frac{1}{x} - \sum_{n=1}^{N} \frac{1}{x+n} + \frac{1}{x-n} = \lim_{x\to 0} \pi cot(\pi x) - \frac{1}{x} = \lim_{x\to 0} \frac{x\pi cos(\pi x) - sin(\pi x)}{x sin(\pi x)}$
We can evaluate this by L'Hôpital's rule, which gives the limit to converge to 0. By the periodicity of $h(x)$, this tell us that that $h(x) = 0$ for all $x \in \mathbb{Z}$.So know, to show that $h(x)$ is identically zero, it suffices to show that $h(x) = 0$ for all $x \in (0, 1)$. Let's define a point of $h(x)$, (x_0, M) to be the point at which $h(x)$ achieves it's maximum value on $[0, 1]$. From the functional equation of $h(x)$, we know that $h(\frac{x_0}{2}) + h(\frac{x_0+1}{2}) = 2h(x_0) = 2M$, however because $h(\frac{x_0}{2}) \leq M$ and $h(\frac{x_0+1}{2}) \leq M$, they must both equal M. A similar argument can be used to show that $\frac{x_0}{2^k} = M$ for all k. however since $\lim_{x\to 0} \frac{x_0}{2^k} = 0$, this tells us that $M = 0$, so $h(x) \leq 0$ for all x. If we multiply both sides of the inequality by negative one, we get $-h(x) \geq 0$, and since h(x) is odd, this is equivalent to $h(-x) \geq 0$ for all x. This tells us that $0 \leq h(x) \leq 0$, or h(x) is identically zero. This thereby shows that $\pi cot(\pi x) = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2x}{x^2-n^2}$.

From the summation form of $\pi cot(\pi x)$, we can find the identity $\pi csc(\pi x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n+x}$.

1.1. **Proof.**

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n+x} = \frac{1}{x} + \sum_{n=1}^{\infty} (\frac{(-1)^n}{x+n} + \frac{(-1)^n}{x-n})$$

$$= \frac{2}{x} + \sum_{n=1}^{\infty} (\frac{2}{x+2n} + \frac{2}{x-2n}) - \frac{1}{x} + \sum_{n=1}^{\infty} (\frac{1}{x+n} - \frac{1}{x-n})$$

$$= \pi cot(\frac{x}{2}\pi) - \pi cot(\pi x) = \pi csc(\pi x)$$

The second step can be justified by seeing that the first sum is equivalent to double the original sum at even integers, and by subtracting the second sum we get a sum which is positive at even integers, and negative at odd ones. This sum is equivalent to our original sum.

Now were finally ready to evaluate the Dirichlet Beta function. Let's start with:
$f(x) = \sum_{k=0}^{\infty} B(2k+1)x^{2k+1} = \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^{2k+1}} x^{2k+1} = \sum_{n=0}^{\infty} (-1)^n \sum_{k=0}^{\infty} (\frac{x}{(2n+1)})^{2k+1}$
Now by summation of geometric series, we get $\sum_{n=0}^{\infty} \frac{\frac{x}{2k+1}}{1-(\frac{x}{2k+1})^2} = \sum_{n=0}^{\infty} \frac{x(2k+1)}{(2k+1)^2-x^2}$. Now we can use partial fraction decomposition to get

$$\frac{x}{2} \sum_{n=0}^{\infty} (-1)^n (\frac{1}{2k+1+x} + \frac{1}{2k+1-x}) = \frac{x}{4} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{n+\frac{1+x}{n}} = \frac{x}{4} \pi csc(\frac{\pi}{2} + \frac{\pi x}{2}) = \frac{\pi x}{4} sec(\frac{\pi x}{2})$$

Now that we have shown that $\frac{\pi x}{4} sec(\frac{\pi x}{2}) = \sum_{k=0}^{\infty} B(2k+1)x^{2k+1}$, let's use the Taylor expansion for $cos(x)$ to get

$$\sum_{k=0}^{\infty} B(2k+1)x^{2k+1} = \frac{\frac{\pi x}{4}}{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}$$

$$\sum_{k=0}^{\infty} B(2k+1)x^{2k} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \frac{\pi}{4}$$

To solve for B(2k+1), let's compare coefficients. The only term that is nonzero will be the x^0 term, which is $B(1)$, so we get that $B(0) = \frac{\pi}{4}$. We can also find that the coefficient of the x^{2m} term is $\sum_{k=0}^{m} B(2k+1)(\frac{\pi}{2})^{2m-2k} \frac{(-1)^{m-k}}{(2m-2k)!} = 0$. Or in other words, $B(2m+1)$ follows the aforementioned recurrence relation with $B(1) = \frac{\pi}{4}$. From this equation, we can derive the values of $B(2m+1)$ needed for the rest of this discussion.

Now it's time to get back to our original problem. We want to find the values of $P, Q, R, \cdots$ from $B(s)$.To do so, let's again use the technique of removing each composite term from B(s). To start, let A = B(s), and
$B = A + \frac{1}{3^s}(A - a)$ and $a = 1$
$C = B + \frac{1}{5^s}(B - b)$ $\quad b = 1 - \frac{1}{3^2}$ $D = C + \frac{1}{7^s}(C - c)$ $\quad c = 1 - \frac{1}{3^2} + \frac{1}{5^s}$ ·
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These values complement to one will approach the value we want, $Z = \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{11^s} - \frac{1}{13^s} - \frac{1}{17^s} + \cdots$. From now on, we will be using decimal representations to seven digits instead of symbolic. Let's start by finding P, so set $s = 3$, giving $A = B(3) \approx 0.9689462$, a = 1, b = 0.9629630, c = 0.9709630, d = 0.96804761 . These values will give us an accurate-enough approximation. So let's continue, with

$B \approx A - \frac{1}{3^3} * 0.0310538 \approx 0.9677961$
$C \approx B - \frac{1}{5^3} * 0.0048331 \approx 0.9677574$
$D \approx C - \frac{1}{7^3} * 0.0032056 \approx 0.9677481$
$D \approx C - \frac{1}{7^3} * 0.0032056 \approx 0.9677481$
$E \approx D - \frac{1}{11^3} * 0, 0.0002995 \approx 0.9677479$

With this many values, we get $P \approx (1 - E) \approx 0.0322521$. Giving $O \approx \frac{1}{2}ln(2) - \frac{1}{3}P =$ 0.3358229.

Let's repeat this process with $s = 5$, which gives $A = B(5) \approx 0.9961578$, with a = 1, b = 0.9958847, c = 0.9962048. Applying the formulas for B and C, we get

$B \approx A - \frac{1}{3^5} * 0.0038422 \approx 0.9961420$
$C \approx B - \frac{1}{5^5} * 0.0002573 \approx 0.9961419$

Thus $Q \approx 1 - C \approx 0.0038581$, and $O \approx \frac{1}{2}ln(2) - \frac{1}{3}P - \frac{1}{5}Q \approx 0.3350513$. Continuing these approximations gives:

$\frac{1}{7}R \approx 0.0000636$
$\frac{1}{9}S \approx 0.0000056$
$\frac{1}{11}T \approx 0.0000005$ $\frac{1}{13}U \approx 0.0000000$

And thus, $O \approx \frac{1}{2}ln(2) - \frac{1}{3}P - \frac{1}{5}Q - \frac{1}{7}R - \frac{1}{9}S - \frac{1}{11}T - \frac{1}{13}U \approx 0.3349816$
So we have discovered that the series $\frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \frac{1}{11} - \frac{1}{13} - \frac{1}{17} + \cdots$ converges to almost exactly 0.3349816. Unfortunately in his original paper, Euler could not find any analytic summation of this series.