DECODING THE ENIGMA OF EQUATIONS: EULER'S RESOLVENTS AND THE QUEST FOR CLOSURE

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Abstract. In this paper, we review Euler's work with Resolvents and his method of solving polynomials using Resolvent Equations. We will show and prove how his method works for Quartic and Cubic Equations, as well as show how his method would work for higher-degree polynomials.

1. Introduction

Throughout history, solving higher-degree polynomial equations has posed a significant challenge for mathematicians worldwide. The pursuit of finding a strict formula or solution for polynomial equations with a degree higher than four has been a persistent problem known as the "Quintic Equation Conundrum". Euler's Resolvent Equation method has proven to play a crucial role in the solving of polynomial equations in general, as well as with the solvability of Quintic Equations. Euler introduced the Resolvent Equation as a tool to transform polynomial equations into more manageable forms, introducing his own method for solving such transformed equations. In this paper, we demonstrate Euler's method using resolvents to solve cubics, quartics and higher-degree polynomials, as well as solving an example cubic.

Definition 1.1. A **polynomial** is an algebraic expression of the form $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where $a_{n-1}, \cdots, a_0$ are real numbers.
Roots - The solutions or zeros of a function $(P(x) = 0)$

Definition 1.2. A **depressed polynomial** is an algebraic expression of the form $P(x) = a_n x^n + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$, where $a_n, \cdots, a_0$ are real numbers and $a_{n-1} = 0$.

Definition 1.3. **Roots** are the solutions or zeros of a function $(P(x) = 0)$.

2. Solving Cubics

2.1. Solving the Depressed Cubic.

An extremely well-known problem throughout history is that of the Depressed Cubic. Originally rumoured to have been solved in 1543, Scipione del Ferro, an established Italian mathematician is credited with first finding a formula to solve a Depressed Cubic. More about the history of the cubic can be found at [Cra12]. A depressed cubic is a cubic of the form

$$x^3 = ax + b$$

where the second-highest term x^2 has a coefficient of 0. This allows Euler to manipulate this equation, taking any root of this equation x and assuming its form as

$$x = \sqrt[3]{A} + \sqrt[3]{B}$$

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where A and B are two roots of some quadratic equation

$$z^2 = \alpha z - \beta \ .$$

This is the **Resolvent Equation** of the Cubic.

Definition 2.1. A **Resolvent Equation** is an equation with 1 degree less ("inferior order" as Euler states) of another polynomial.

Based on Vieta's Equations, we get

$$\alpha = A + B, \ \ \beta = AB \ .$$

And after squaring the first equation $x = \sqrt[3]{A} + \sqrt[3]{B}$ to get

$$x^3 = A + B + 3\sqrt[3]{AB}\left(\sqrt[3]{A} + \sqrt[3]{B}\right)$$

and substituting $x = \sqrt[3]{A} + \sqrt[3]{B}$, we end up with the cubic equation

$$x^3 = A + B + 3x\sqrt[3]{AB} \ .$$

Comparing this equation with the original depressed cubic equation $x^3 = ax + b$, it is shown that

$$a = 3\sqrt[3]{AB} = 3\sqrt[3]{\beta}, \ \ b = A + B = \alpha \ .$$

And by reversing these equations

$$\alpha = b, \ \ \beta = a^3/27 \ .$$

Finally, after substituting this into the equation $z^2 = \alpha z - \beta$, we have the modified resolvent equation

$$z^2 = bz - a^3/27 \ .$$

Reverting back to the original equation $x = \sqrt[3]{A} + \sqrt[3]{B}$ and using the fact that any cube root of a quantity has a triple value, Euler expressed the other 2 roots in the following form

$$x = \mu\sqrt[3]{A} + \sigma\sqrt[3]{B}$$

where $\mu\sigma = 1$. The reason we are doing this is that the cube roots of A and B respectively can have 9 possible combinations, thus in order to find the 2 definitive other roots, we take μ and σ as 2 **roots of unity** to find the other 2 roots.

Definition 2.2. A **root of unity** is a complex number, which when raised to the power of any integer n results in a value equal to 1. More about this can be read at [AKT14].

Thus μ and σ consequentially possess the following values in any order as they are the roots of unity:

$$\frac{-1 + \sqrt{-3}}{2}, \ \ \frac{-1 - \sqrt{-3}}{2} \ .$$

Therefore besides the root $x = \sqrt[3]{A} + \sqrt[3]{B}$, the other roots are, in no particular order

$$x = \frac{-1 + \sqrt{-3}}{2}\sqrt[3]{A} \ + \ \frac{-1 - \sqrt{-3}}{2}\sqrt[3]{B}$$

and

$$x = \frac{-1 - \sqrt{-3}}{2}\sqrt[3]{A} \ + \ \frac{-1 + \sqrt{-3}}{2}\sqrt[3]{B} \ .$$

2.2. **Converting a Regular Cubic to a Depressed Cubic.** Taking any cubic in the form $ax^3 + bx^2 + cx + d = 0$ and dividing by a to make the Cubic **monic**, we get

$$x^3 + \frac{b}{a}x^2 + \frac{c}{a}x + \frac{d}{a} = 0\ .$$

Then substituting $x = y - \frac{b}{3a}$ into the equation (also known as the **Tschirnhaus Transformation**), we get:

$$\left(y - \frac{b}{3a}\right)^3 + \frac{b}{a}\left(y - \frac{b}{3a}\right)^2 + \frac{c}{a}\left(y - \frac{b}{3a}\right) + \frac{d}{a} = 0$$

Definition 2.3. The **Tschirnhaus Transformation** is the substitution required to turn a polynomial into a depressed polynomial eg. $ax^3 + bx^2 + cx + d = 0 \to py^3 + qy + r = 0$ where $y = x - \frac{b}{ka}$. More can be found at [BR99].

After further simplifying (skipping a few steps) we get the Depressed Cubic Equation in terms of y:

$$y^3 + \left(\frac{c}{a} - \frac{b^2}{3a^2}\right)y + \left(\frac{2b^3}{27a^3} - \frac{bc}{3a^2} + \frac{d}{a}\right) = 0$$

And from this equation, we can see that we have converted a regular cubic into a depressed one while still using the original coefficients in our manipulation.

2.3. **Cubics - An Example.** Let us take an example to demonstrate Euler's method; the depressed cubic equation

$$x^3 = 7x - 6$$

where there is no quadratic term and we apply Euler's method to it. Firstly, we take any one root x and represent it as

$$x = \sqrt[3]{A} + \sqrt[3]{B}$$

where A and B are two roots of some quadratic equation

$$z^2 = \alpha z - \beta$$

where $\alpha = b = -6$ and $\beta = a^3/27 = 7^3/27$ as we derived earlier. Thus the new formula is

$$z^2 = -6z - 7^3/27$$

or, in another form,

$$z^2 + 6z + 7^3/27 = 0$$

which after finding, we see the roots are

$$z = -3 + \frac{10i}{9}\sqrt{3}\ or\ z = -3 - \frac{10i}{9}\sqrt{3}$$

or

$$z = -3 + \sqrt{-\frac{100}{27}}\ or\ z = -3 - \sqrt{-\frac{100}{27}}\ .$$

Taking $-3+\sqrt{-\frac{100}{27}}$ as A and $-3-\sqrt{-\frac{100}{27}}$ as B and one of the roots as $x = \sqrt[3]{A}+\sqrt[3]{B} = 2$, the other roots are, in no particular order

$$x = \frac{-1+\sqrt{-3}}{2}\sqrt[3]{A}\ +\ \frac{-1-\sqrt{-3}}{2}\sqrt[3]{B}$$

or

$$x = \frac{-1-\sqrt{-3}}{2}\sqrt[3]{A} \ + \ \frac{-1+\sqrt{-3}}{2}\sqrt[3]{B}$$

, or:

$$x = \frac{-1+\sqrt{-3}}{2}\sqrt[3]{-3+\sqrt{-\frac{100}{27}}} \ + \ \frac{-1-\sqrt{-3}}{2}\sqrt[3]{-3-\sqrt{-\frac{100}{27}}}$$

or

$$x = \frac{-1-\sqrt{-3}}{2}\sqrt[3]{-3+\sqrt{-\frac{100}{27}}} \ + \ \frac{-1+\sqrt{-3}}{2}\sqrt[3]{-3-\sqrt{-\frac{100}{27}}} \ .$$

After simplifying these expressions using the **principal roots** of the equation, we get

$$x = -3 \text{ or } x = 1 \ .$$

Thus, we have proven that $x^3 = 7x - 6$ has the roots $x = 1, 2$ or -3 using Euler's method for solving polynomial equations using resolvents.

2.4. **Recovering the Original Roots.** Using the Tschirnhaus Transformation, we were able to turn a regular cubic into a depressed one. This allowed us to use Euler's method and solve the depressed resultant cubic. In Euler's method, we first represent one root of the equation x^3 as the addition of 2 cube roots $\sqrt[3]{A}$ and $\sqrt[3]{B}$, where A and B were roots of a regular quadratic. Through manipulations and substitutions of these roots and the coefficients of the Resolvent Equation (α and β), we were able to use the triple value of the cube root to find the other 2 roots of the depressed cubic equation. One could simply reverse the Tschirnhaus Transformation of $x = y - \frac{b}{3a}$ by subtracting $\frac{b}{3a}$ from each of the roots of the depressed cubic to get the roots of the regular cubic ie.

$$p,\ q,\ r \ \mapsto\ p - \frac{b}{3a},\ q - \frac{b}{3a},\ r - \frac{b}{3a} \ .$$

3. Solving Quartics

3.1. **Solving the Depressed Quartic.** Quartic equations have been a subject of fascination and study in mathematics for centuries. Just like the famous depressed cubic problem, which attracted the attention of mathematicians in the past, quartic equations have their own set of challenges and intriguing properties. The search for a general formula to solve quartic equations has been an ongoing endeavor, with mathematicians throughout history making significant contributions. However unlike equations to the fifth power or above, there are several methods that can be used to solving quartics. Exploring the history and solutions of quartic equations can provide valuable insights into the development of algebraic techniques and the evolution of mathematical thinking. More about quartics can be viewed at [Mer92].

Using a similar approach to the above, we first take the general form of a depressed quartic equation as

$$x^4 = ax^2 + bx + c$$

and one of the roots as

$$x = \sqrt{A} + \sqrt{B} + \sqrt{C}$$

where A, B, and C are all roots of the Resolvent Equation

$$z^3 = \alpha z^2 - \beta z + \gamma \ .$$

From Vieta's Equations, we get

$$\alpha = A + B + C, \quad \beta = AB + BC + AC, \quad \gamma = ABC\ .$$

In an attempt to free the equation $x = \sqrt{A} + \sqrt{B} + \sqrt{C}$ from irrationality, we square it to get

$$x^2 = A + B + C + 2\sqrt{AB} + 2\sqrt{AC} + 2\sqrt{BC}$$

and by subtracting α, we get:

$$x^2 - \alpha = 2\sqrt{AB} + 2\sqrt{AC} + 2\sqrt{BC}\ .$$

By squaring it again

$$x^4 - 2\alpha x^2 + \alpha^2 = 4AB + 4AC + 4BC + 8\sqrt{ABC}\left(\sqrt{A} + \sqrt{B} + \sqrt{C}\right) = 4\beta + 8x\sqrt{\gamma}$$

and simplifying, we get

$$x^4 = 2\alpha x^2 + 8x\sqrt{\gamma} + 4\beta - \alpha^2\ .$$

Comparing this to $x^4 = ax^2 + bx + c$, we get

$$a = 2\alpha, \quad b = 8\sqrt{\gamma}, \quad c = 4\beta - \alpha^2$$

which shows that

$$\alpha = \frac{a}{2}, \quad \gamma = \frac{b^2}{64}, \quad \beta = \frac{c}{4} + \frac{a^2}{16}\ .$$

Therefore by subbing these into the cubic equation with roots A, B, C, we get

$$z^3 = \frac{a}{2}z^2 - \frac{4c + a^2}{16}z + \frac{b^2}{64}\ .$$

Apart from $x = \sqrt{A} + \sqrt{B} + \sqrt{C}$, Euler takes the 3 other solutions to be of the form

$$\sqrt{A} - \sqrt{B} - \sqrt{C}, \ \sqrt{B} - \sqrt{A} - \sqrt{C}, \ \sqrt{C} - \sqrt{A} - \sqrt{B}\ .$$

Substituting in $z = \sqrt{t}$, we get

$$\left(t + \frac{4c + a^2}{16}\right)\sqrt{t} = \frac{at}{2} + \frac{b^2}{64}\ .$$

And after a series of expansions and simplifying,

$$t^3 = \left(\frac{a^2}{8} - \frac{c}{2}\right)t^2 + \left(\frac{ab^2}{64} - \frac{c^2}{16} - \frac{a^2c}{32} - \frac{a^4}{256}\right)t + \frac{b^3}{4096}\ .$$

After this final result, it is visible that this equation has the property that its roots are the squares of the roots (A, B and C) of the prior equation. Despite this method being tedious, we have nevertheless obtained a cubic equation from a quartic one and found the roots alike.

3.2. **Converting a Regular Quartic to a Depressed Quartic.** Taking any quartic in the form $ax^4 + bx^4 + cx^2 + dx + e = 0$ and dividing by a to make the Quartic monic, we get

$$x^4 + \frac{b}{a}x^3 + \frac{c}{a}x^2 + \frac{d}{a}x + \frac{e}{a} = 0 \; .$$

Then substituting $x = y - \frac{b}{4a}$ into the equation, we get

$$\left(y - \frac{b}{4a}\right)^4 + \frac{b}{a}\left(y - \frac{b}{4a}\right)^3 + \frac{c}{a}\left(y - \frac{b}{4a}\right)^2 + \frac{d}{a}\left(y - \frac{b}{4a}\right) + \frac{e}{a} = 0 \; .$$

After further simplifying (skipping a few steps) we get the Depressed Cubic Equation in terms of y:

$$y^4 + \left(\frac{6b}{a}\right)y^2 + \left(\frac{b^2 - 4ac}{a^2}\right)y + \frac{-3b^3 + 4abcd - 8a^{2e}}{a^3} = 0 \; .$$

From this equation, we can see that we have converted a regular quartic into a depressed one while still using the original coefficients in our manipulation.

3.3. **Recovering the Original Roots.** Using the substitution of $x = y - \frac{b}{4a}$, we were able to turn a regular quartic into a depressed one. This allowed us to use Euler's method and solve the depressed resultant quartic. In Euler's method, we first represent one root of the equation x^4 as the addition of 3 square roots $\sqrt{A}$, $\sqrt{B}$ and $\sqrt{C}$, where A, B and C were roots of a regular cubic. Using other manipulations, we found the other 3 roots. One could simply reverse the substitution of $x = y - \frac{b}{4a}$ by subtracting $\frac{b}{4a}$ from each of the roots of the depressed quartic to get the roots of the regular quartic ie.

$$p,\; q,\; r,\; s \mapsto\; p - \frac{b}{4a},\; q - \frac{b}{4a},\; r - \frac{b}{4a},\; s - \frac{b}{4a} \; .$$

4. Solving Quintics and Higher Degree Polynomials

4.1. **General Approach.** In General, the approach to all methods is the same; first represent 1 root in terms of the root of the polynomial's resolvent equation. Then perform some manipulations and substitutions to end up with 2 monic resolvent equations, on which the values of the other roots can be solved from a system of equations.

4.2. **Method to Solving the Depressed Equation.** Using a similar approach to the above, we first take the general form of a depressed quintic equation as

$$x^5 = ax^3 + bx^2 + cx + d$$

whose Resolvent Equation would be as follows

$$z^4 = \alpha z^3 - \beta z^2 + \gamma z - \sigma \; .$$

In general, for a polynomial with degree n, the equation would be

$$x^n = ax^{n-2} + bx^{n-3} + cx^{n-4} + \;...$$

with Resolvent Equation

$$z^{n-1} = \alpha z^{n-2} - \beta z^{n-3} + \gamma z^{n-4} - \sigma z^{n-5} + \;...$$

whose roots are known with values

$$x = \sqrt[n]{A} + \sqrt[n]{B} + \sqrt[n]{C} + \;...$$

4.3. **Problem.** Here is where the problem occurs with polynomials with a degree higher than 4. No Resolvent Equation can be derived for such polynomials and Euler explicitly states "*Although if the given equation has more than four dimensions I am so far not able to define a resolvent equation*". With famous mathematicians after Euler's time also proving that it is not possible to find the resolvent equation of a quintic, this is where the problem comes to a standstill. Some well-known papers working with the concept of quintics include the **Abol-Ruffini Theorem** ([Żoł00]) and **Galois Theory** ([CD20]). However, if someone were to theoretically figure out the resolvent equation of a quintic, he or she would be able to apply the same steps shown above to solve for the roots of the quintic. But until then, Euler's method is the closest we have to finding a definitive solution for any polynomial with degree 5 or higher.

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