Several Considerations about Hypergeometric Series

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1. Brief History

Euler's work on the hypergeometric series is related to his contributions to mathematical analysis, particularly in the area of special functions and infinite series $\sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)^3}$. In math, the term "hypergeometric series" was first used by John Wallis in 1655 in his book "Arithmetica Infinitorum". Hypergeometric series were studied by Leonhard Euler and Gauss in the 18th and 19th centuries respectively. Moreover in the 19th century Ernst Kummer and Bernhard Riemann by means of the differential equation it satisfies. This did not scant to the contributions of numerous other people in mathematics, ranging time from the 17th century to the early 19th century. This time has provided many important ideas and motivations for more sophisticated developments.

2. Introduction

Consider the following geometric series,

$$1 + x + x^2 + x^3 \cdots = (1-x)^{-1}$$

where the first term is called constant second term is linear and so on. As mentioned earlier term "hypergeometric" was first used by John Wallis. He extended ordinary geometric series to the hypergeometric series in his book *Arithmetica Infinitorum*.

$$1 + a + a(a+b) + a(a+b)(a+2b) + \dots, \tag{1}$$

where nth term is,

$$a(a+b)(a+2b)\dots(a+(n-1)b), \tag{2}$$

which became with $b = 1$, the Pochammer symbol.

$$(a)_n = a(a+1)(a+2)\dots(a+n-1) = \prod_{k=1}^{n}(a+k-1), \tag{3}$$

n is a non-negative integer. It is named after mathematician Leo Augustus Pochammer (1841-1920), his usage of notation was for the rising factorial.

3. Hypergeometric Function

Euler first introduced the power series expansion but in a different form than most of the other forms of power series. Let a, b, c denote functions, hypergeometric functions had defined as,

$$1 + \frac{ab}{x}\frac{z}{1!} + \frac{a + a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \cdots = F(a, b, c, z), \tag{4}$$

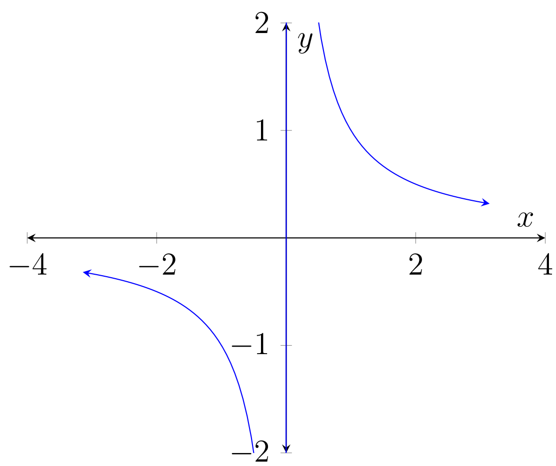
where F is the hypergeometric function. The function takes the head amongst the standard mathematical functions used in both pure and applied mathematics. It is the kind of series that gives us many different series depending on the a, b, c.

- If $a = 1$ and $b = c$ or three of them are equal to 1 then we get a geometric series as we showed in (3).
- If b and c are equal to 1 then it becomes the known binomial series in the form $(1-z)^{-a}$
- If $a = b = \frac{1}{2}$ and $c = \frac{3}{2}$ gives us $\frac{\arcsin(\sqrt{z}}{\sqrt{z}}$
- If again a and b is equal to 1, c is equal to 2 and $z = 1$ will give us the harmonic series. Moving to common notation for hypergeometric series

$$ {}_1F_1(1, 1, 2; 1) = \sum_{n=0}^{\infty} \frac{(1)_n (1)_n 1^n}{(2)_n n!}, \tag{5}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n+1} = \sum_{n=1}^{\infty} 1 + \frac{1}{2} + \frac{1}{4} + \ldots . \tag{6}$$

We pretty much know why it is called "harmonic" because consider the function in the form $F = \frac{1}{x}$. If you graph it,



(7)

- If you were to draw squares that have have width 1 and height decreases as x goes to ∞ but it will always have widths 1 and it is the harmony there like a ladder but every time you take a step it you will move horizontally less. You can also consider as wavelengths which gives a better intangible comparison.

To mention some important developments using hypergeometric series, Gauss has announced *text* in Latin. At Paris International Congress of Mathematicians in 1812, but as we mentioned earlier if he were to just show about Euler's hypergeometric series it would not make much noise but he showed the hypergeometric functions as solution to second order ordinary differential equation through out the complex plane. He also showed that the series converges for absolute value of z is less than 0 and divergent for absolute value of z greater than 1.
Gauss preferred the series that are represented by hypergeometric functions as functions with 4 variables but not as 1 variable and 3 parameters a, b, c. From the British Mathematician Barnes we know that modern notation is,

$$ {}_2F_1(a, b; c; z). \tag{8} $$

We showed some series using hypergeometric series, implying that if we look in more detail we might get some more interesting stuff as well.

- $e^z = {}_1F_1(1; 1; z)$
- $\sinh(z) = z.{}_0F_1(\frac{3}{2}; \frac{z^2}{4})$
- $\cosh(z) = {}_0F_1(\frac{1}{2}; \frac{z^2}{4})$
- $\frac{\arcsin(z)}{z} = {}_2F_1(\frac{1}{2}, \frac{2}{3}, \frac{3}{2}; -\frac{27x^2}{4})$
- ${}_2F_1(1, 1; 2; -z) = \frac{\ln(1+z)}{z}$

We learned the hyperbolic functions and Euler's formula for it so we can conclude that we can easily express some rational functions as hypergeometric functions as shown and even some logarithmic functions. To give a more precise definition to hypergeometric functions,

$$ {}_qF_p(\begin{pmatrix} a_1, & a_2, \ldots, & a_p \\ b_1, & b_2, \ldots, & b_q \end{pmatrix}; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \ldots (a_q)_k}{(b_1)_k \ldots (b_p)_k} \frac{z^k}{k!}. \tag{9} $$

Where series takes q numerator and p denominator arguments.

4. More in Hypergeometric Series

It is got attention by Sriniava Rao(Srinivasa Rao K 1981 Comp. Phys. Commun 22 297). He showed that hypergeometric series has it its own folded version, where we built the series by factoring in by in every time, it is as followed,

$$ 1 + \frac{ab}{c}\frac{z}{1}\left(1 + \frac{a(a+1)b(b+1)}{(c+1)}\frac{z}{2}\left(1 + \frac{(a+2)(b+2)}{c+2}\frac{z}{3}\right)1 + \ \ldots\right). \tag{10} $$

This form is really useful for computation because as seen whenever we have negative numerators it becomes a polynomial in degree n. For example plug $b = -2$ to (4), plugging in it we will get,

$$ 1 + \frac{ab}{c}\frac{z}{1!} + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} = 1 - \frac{2a}{c}\frac{z}{1!} + \frac{2a(a+1)}{c(c+1)}\frac{z^2}{2!}. \tag{11} $$

As we seen in Euler's work and on some specific series that has Horner's Scheme, that is is really useful in terms of finding roots of polynomial and its combinatoric applications.
In 1748 Euler(Euler L 1748 Introduction to Analysis Infinitorum vol I (Lausanne: Bousquet)) got the famous transformation for hypergeometric series,

$$ {}_2F_1(a,-n;c;z) = (1-z)^{c+n-a}{}_2F_1(c-a,c+n;c;z). \tag{12}$$

Vandormende found the theorem called Chu-Vandermonde identity which is extension version of binomial theorem for $z=1$.

$$ {}_2F_1(a,-n;c;1) = \frac{(c-a)(c-a+1)(c-a+2)\dots(c-a+n-1)}{c(c+1)(c+2)\dots(c+n-1)} \tag{13}$$

we can also write this simply in the most known form,

$$ {}_2F_1(a,-n;c;1) = \frac{(c-a)_n}{(c)_n}. \tag{14}$$

This is actually just Gauss's hypergeometric theorem but special case of it. In (14)

$$(a)_n = a(a+1)(a+2)\dots(a+n-1). \tag{15}$$

This can be written in terms of *Gamma Function*, which is just $(n-1)!$ meaning that $n\Gamma(n) = n!$. We also recalled Pochhammer symbol in there. Gauss also proved the summation formula of his for Chu-Vandermonde theorem.

$$ {}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = \frac{(c-a)_n}{(c)_n}. \tag{16}$$

Only difference is that $b=-n$. The theorem of Gauss is just consequence of Euler's work, which is integral definition of hypergeometric function. It is not really weird since Gauss pretty much worked on same stuff with Euler and extended it when he could.

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^\infty t^{b-1}(1-t)^{a-c}(1-tz)^{-a}dt. \tag{17}$$

Considering $z=1$ this will become famous beta function, which is called Euler Integral that is a function known as close relationship with gamma function with binomial coefficients. Defined as follows,

$$B(z_1,z_1) = \int_0^1 t^{z_1-1}(1-t)^{z_2-1}dt. \tag{18}$$

Where z_1, z_2 are complex numbers. After (17) we get for $z=1$,

$$\int_0^1 t^{b-1}(1-t)^{c-a-b-1}dt = B(b,c-a-b) = \frac{\Gamma(b)\Gamma(c-a-b)}{\Gamma(c-a)} \tag{19}$$

We gave the definition at (18) and if we substitute this into (19) we get the Gauss summation formula (17).

5. Bibliography

References

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