Coefficients of powers of $1+x+x^2+\cdots+x^k$

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It is very well known that

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n = \sum_{k=0}^{n}\binom{n}{k}x^k.$$

Euler was curious about the coefficients of the expansion of the expression $(1+x+x^2+x^3+\cdots+x^k)^n$ for some k. Throughout this paper, we will use the notation $\binom{n}{i}_{k-1}$ to represent the coefficient of the term x^i in the expansion of $(1+x+x^2+x^3+\cdots+x^k)^n$ (in particular, it is possible that $i > n$).

We first look at $(1+x+x^2)^n$. Let's compute some expansions for small n.

$$(1+x+x^2)^0 = 1,$$
$$(1+x+x^2)^1 = 1+x+x^2,$$
$$(1+x+x^2)^2 = 1+2x+3x^2+2x^3+x^4,$$
$$(1+x+x^2)^3 = 1+3x+6x^2+7x^3+6x^4+3x^5+x^6,$$
$$(1+x+x^2)^4 = 1+4x+10x^2+16x^3+19x^4+16x^5+10x^6+4x^7+x^8,$$
$$(1+x+x^2)^5 = 1+5x+15x^2+30x^3+45x^4+51x^5+45x^6+30x^7+15x^8+5x^9+x^{10}.$$

There's no obvious patterns in the coefficients, asides from the fact that they're symmetric. This makes sense, since the the polynomial we're raising to a power is symmetric. This gives us that $\binom{n}{i}_3 = \binom{n}{2n-i}_3$, although this isn't particularly useful for determining a pattern for the coefficients.

What Euler did now was write the expression as $[1+x(1+x)]^n$. We can then use the binomial theorem on this to obtain

$$1+\binom{n}{1}x(1+x)+\binom{n}{2}x^2(1+x)^2+\cdots+\binom{n}{n}x^n(1+x)^n=\sum_{i=0}^{n}\binom{n}{i}x^i(1+x)^i.$$

We can now express $(1+x)^i$ using the binomial theorem, giving is

$$\sum_{i=0}^{n}\binom{n}{i}x^i\sum_{j=0}^{i}\binom{i}{j}x^j=\sum_{i=0}^{n}\sum_{j=0}^{i}\binom{n}{i}\binom{i}{j}x^{i+j}.$$

Now we can determine $\binom{n}{c}_3$. We need $c=i+j$, so we have that

$$\binom{n}{c}_3=\sum_{c=i+j}\binom{n}{i}\binom{i}{j}.$$

For example, we know from our manual calculation of some of these polynomials above that $\binom{4}{4}_3=19$. Using our expression above, we get that

$$\binom{4}{4}_3=\sum_{4=i+j}\binom{4}{i}\binom{i}{j}=\binom{4}{0}\binom{0}{4}+\binom{4}{1}\binom{1}{3}+\binom{4}{2}\binom{2}{2}+\binom{4}{3}\binom{3}{1}+\binom{4}{4}\binom{4}{0}$$

$$=0+0+6+4\cdot3+1=19.$$

To be sure, below is a table for every $\binom{n}{i}_3$ up till $n=3$.

$\binom{0}{0}_3=1$						
$\binom{1}{0}_3=1$	$\binom{1}{1}_3=1$	$\binom{1}{2}_3=1$				
$\binom{2}{0}_3=1$	$\binom{2}{1}_3=3$	$\binom{2}{2}_3=3$	$\binom{2}{3}_3=2$	$\binom{2}{4}_3=1$		
$\binom{3}{0}_3=1$	$\binom{3}{1}_3=3$	$\binom{3}{2}_3=6$	$\binom{3}{3}_3=7$	$\binom{3}{4}_3=6$	$\binom{3}{5}_3=3$	$\binom{3}{6}_3=1$

These match up with the polynomials computed earlier, so we know we're on the right path.

Now let's look at $(1+x+x^2+x^3)^n$. We use the same strategy as before, writing the expression at $[1+x(1+x+x^2)]^n$. First expanding using the binomial theorem gives

$$\sum_{i=0}^{n}\binom{n}{i}x^i(1+x+x^2)^i.$$

Now we can expand the inside using our result for $k=2$. We obtain

$$\sum_{i=0}^{n}\binom{n}{i}x^i\left[\sum_{j=0}^{i}\sum_{k=0}^{j}\binom{i}{j}\binom{j}{k}x^{j+k}\right]=\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j}\binom{n}{i}\binom{i}{j}\binom{j}{k}x^{i+j+k}.$$

Thus, we have that

$$\binom{n}{c}_4=\sum_{c=i+j+k}\binom{n}{i}\binom{i}{j}\binom{j}{k}x^{i+j+k}.$$

This is a closed form, but we have three binomials inside a sum that has $\binom{c+i+j+k}{2}$ terms, which can big fast. Not to mention, the closed form looks kind of ugly with all the binomials. So let's try to find a different closed form. In fact, we don't have to try hard to find a much cleaner closed form. Go back to this sum:

$$\sum_{i=0}^{n}\binom{n}{i}x^i(1+x+x^2)^i.$$

Instead of expanding the trinomial all the way, we can expand it as follows:

$$\sum_{i=0}^{n}\binom{n}{i}x^i\sum_{j=0}^{2i}\binom{i}{j}_3x^j=\sum_{i=0}^{n}\sum_{j=0}^{2i}\binom{n}{i}\binom{i}{j}_3x^{i+j}.$$

From this, it easily follows that

$$\binom{n}{c}_4=\sum_{c=i+j}\binom{n}{i}\binom{i}{j}_4.$$

Let's look at $k=5$ before we tackle the general case. We can write the expression as $[1+x(1+x+x^2+x^3)]^n$. Expanding using the binomial theorem yields

$$\sum_{i=0}^{n}\binom{n}{i}x^i(1+x+x^2+x^3)^i.$$

Then using the result for $k=4$ gives

$$\sum_{i=0}^{n}\binom{n}{i}x^i\sum_{j=0}^{3i}\binom{i}{j}_4x^j=\sum_{i=0}^{n}\sum_{j=0}^{3i}\binom{n}{i}\binom{i}{j}_4x^{i+j}.$$

From this is follows that

$$\binom{n}{c}_5=\sum_{c=i+j}\binom{n}{i}\binom{i}{j}_4.$$

We can now prove the general result using induction. We already showed the base case $k = 3$. Now assume for some k we have that

$$\binom{n}{c}_k = \sum_{c=i+j} \binom{n}{i}\binom{i}{j}_{k-1}.$$

To show the result for $k+1$, we expand $[1 + x(1 + x + \cdots + x^{k-1})]^n$, which yields

$$\sum_{i=0}^{n} \sum_{j=0}^{(k-1)i} \binom{n}{i}\binom{i}{j}_k x^{i+j},$$

which yields

$$\binom{n}{c}_{k+1} = \sum_{c=i+j} \binom{b}{i}\binom{i}{j}_k.$$

Extending Euler's Results

We can easily extend Euler's results by considering the inside polynomial with arbitrary coefficients. For example, if we look at $(a + bx + cx^2)^2$ and rewrite it using the same method as for the original trinomial, we obtain

$$\sum_{i=0}^{n} \binom{n}{i} a^{n-i} x^i (b + cx)^i.$$

Upon expanding the inside using the binomial theorem and rearranging yields

$$\sum_{i=0}^{n} \sum_{j=0}^{i} \binom{n}{i}\binom{i}{j} a^{n-i} b^{i-j} c^j x^{i+j}.$$

Now, if we're looking for the coefficient of x^k, we obtain that the coefficient is

$$\sum_{k=i+j} \binom{n}{i}\binom{i}{j} a^{n-i} b^{i-j} c^j.$$

As ugly as this looks, it does extend Euler's result to an arbitrary polynomial. The case $a = b = c = 1$ was Euler's original result.

In fact, there's a much easier way to derive these results using the multinomial theorem. By the multinomial theorem, the coefficient of x^m for the general trinomial is

$$\sum_{m=i+j+k} \binom{n}{i,\, j,\, k} a^i b^j c^k.$$

Note that in our original formula, we have

$$\binom{n}{i}\binom{i}{j} = \frac{n!}{i!(n-i)!}\frac{i!}{j!(i-j)!} = \frac{n!}{(n-i)!(i-j)!j!} = \binom{n}{n-i,\, i-j,\, j},$$

so the two forms are essentially equivalent.

In particular, applying the multinomial theorem to an arbitrary polynomial $a_0 + a_1 x + \cdots + a_k x^k$ to the nth power yields that the coefficient of x^m is

$$\sum_{m=b_0+b_2+\cdots+b_k} \binom{n}{b_0,\, b_1,\, \cdots,\, b_k} a_0^{b_0} a_1^{b_1} \cdots a_k^{b_k}.$$

References

[1] Euler, Leonhard. "On the expansion of the power of any polynomial $1 + x + x^2 + x^3 + x^4 +$ etc." arXiv preprint math/0505425 (2005).