ON DIOPHANTINE EQUATIONS OF THE QUARTIC FORMS

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We report the investigation for the general solutions to diophantine equations of form $2a^4 - b^4 = c^2$ and $3a^4 + b^4 = c^2$. As a bonus, we also discuss Euler's conjecture on the diophantine $a^4 = b^4 + c^4 + d^4$, which was disproved by Elkies (1988).

1. Introduction

This section will discuss Euler's creation of general methodology in approaching quartic diophantines that equal to a square. Of note, see the following equation:

$$a^2x^4 + 2abx^3y + cx^2y^2 + 2bdxy^3 + d^2y^4 = N^2$$

So noting the terms of a^2x^4, d^2y^4, and the other terms excluding the middle, we rewrite as

$$(ax^2 + bxy + dy^2)^2 + (c - b^2 - 2ad)x^2y^2 = N^2$$

For simplicity let $c - b^2 - 2ad = mn$, $ax^2 + bxy + dy^2 = \lambda(mp^2 - nq^2)$, $xy = 2\lambda pq$.
Remarkably, we end with

$$\lambda^2(mp^2 - nq^2)^2 + mn4\lambda^2p^2q^2 = \lambda^2(m^2p^4 + n^2q^4 + 2mnp^2q^2$$

$$\lambda^2(mp^2 + nq^2)^2 = N^2$$

Surely, we end up with a square.

We are permitted to let $x = 2\lambda pq$ with unity in place of y, IF we allow fractions, such that

$$ax^2 + bxy + dy^2 = \lambda(mp^2 - nq^2)$$

$$4\lambda^2ap^2q^2 + 2\lambda bpq + d = \lambda mp^2 - \lambda nq^2$$

This is a quadratic equation, and we can calculate the roots for p and q.

$$p = \frac{-\lambda bq \pm \sqrt{\lambda md + \lambda^2q^2(b^2 - 4ad + mn) - 4\lambda^3naq^4}}{4\lambda^2q^2a - \lambda m}$$

$$q = \frac{-\lambda bp \pm \sqrt{-\lambda nd + \lambda^2p^2(b^2 - 4ad + mn) + 4\lambda^3naq^4}}{4\lambda^2p^2a + \lambda n}$$

With λ of so many possibilities, Euler proves that there must exist such a value where the square root exists. Then, also say p and q were found. We construct some series as follows, to find other roots:

$$p + p' = -\frac{2bq}{4\lambda aq^2 + m}$$
$$q + q' = -\frac{2bp}{4\lambda ap^2 + n}$$

Thus, Euler avoided finding values of other values tediously as long as some solution was found for the diophantine. The following series follows:

$$p' = \frac{-2bq}{4\lambda aq^2 - m} - p; \quad q' = \frac{-2bp'}{4\lambda ap'^2 + n} - q$$
$$p'' = \frac{-2bq'}{4\lambda aq'^2 - m} - p'; \quad q' = \frac{-2bp''}{4\lambda ap''^2 + n} - q'$$
$$\cdots$$

And even better, this works for series of $(q, p), (q', p'), (q'', p'') \cdots$ Thus:

$$x = 2\lambda pq, 2\lambda qp', 2\lambda p'q' \cdots$$

OR,

$$2\lambda qp, 2\lambda pq', \cdots$$

under assumption that $y = 1$. But any fractions that arise here shouldn't be a concern since we may assign the denominator for y, numerator for x.

Now, let $y = 1$, and solve for the following quartic equation:

$$a^2x^4 + 2abx^3 + cx^2 + 2bdx + d^2 = (ax^2 + bx - d)^2$$

Then,

$$x = \frac{4bd}{b^2 - 2ad - c} = \frac{-4bd}{mn + 4ad}, \quad c = mn + b^2 + 2ad$$
$$\frac{ax^2 + bx + d}{x} = A, \quad mp^2 - nq^2 = 2Apq$$
$$\frac{p}{q} = \frac{A + \sqrt{A^2 + mn}}{m}$$

where we can extract all the roots.

Such formulas can be reduced effectively given the following situation, though:

$$\alpha A^4 \pm \beta B^4 = N^2$$

Let $C = \frac{1+x}{1-x}$ and $\alpha + \beta = a^2$ given $\frac{A}{B} = C, \alpha C^4 \pm \beta = N^2$, which then $\alpha \pm \beta = N^2$ by letting $C = 1$. Then,

$$a^2 + 4(\alpha - \beta)x + 6a^2x^2 + 4(\alpha - \beta)x^3 + a^2x^4$$

So this is the general methodology of Euler.

2. The First Form: $2A^4 - B^4 = N^2$

Here, this case would be that $2C^4 - 1 = N^2$, so $\alpha = 2, \beta = -1, \alpha + \beta = 1 = a^2, a = 1$. Thus we have that

$$1 + 12x + 6x^2 + 12x^3 + x^4 = N^2$$
$$(1 + 6x + x^2)^2 - 32x^2 = N^2$$

So

$$1 + 6x + x^2 = \lambda(p^2 + 2q^2) \text{ and } 4x = 2\lambda pq$$

Then $x = \frac{1}{2}\lambda pq$ and to avoid fractions say $q = 2q$,

$$1 + 6\lambda pq + \lambda^2 p^2 q^2 = \lambda p^2 + 8\lambda q^2$$

$$p = \frac{-3\lambda q \pm \sqrt{8\lambda^3 q^4 + \lambda}}{\lambda^2 q^2 - \lambda}, \quad q = \frac{-3\lambda p \pm \sqrt{8\lambda^3 p^4 + \lambda}}{\lambda^2 p^2 - 8\lambda}$$

$$p + p' = \frac{-6q}{\lambda q^2 - 1} \text{ and } q + q' = \frac{-6p}{\lambda p^2 - 8}$$

And thus we can construct the following series where:

$$q' = \frac{-6p}{p^2 - 8} - q; \;\; p' = \frac{-6q'}{q'^2 - 1} - p$$
$$q'' = \frac{-6p'}{p'^2 - 8} - q'; \;\; p'' = \frac{-6q''}{q''^2 - 1} - p'$$
$$\cdots$$

Thus, by letting $(q, p) = (0, 1)$ we get the following cases:

$$0; 1; \frac{6}{7}; \frac{239}{19}; \cdots$$

Then we take these values and multiply neighboring pairs to get x. For instance, $0 \times 1, 1 \times \frac{6}{7}, \cdots$.

$$x = 0, \frac{6}{7}, \frac{1434}{91}, \cdots, \text{ and } C = 1, 13, \frac{-1525}{1343}$$

We can in fact use other values in the series to let $(q, p) = (1, \frac{6}{7})$ and evaluate different series. And thus we can find the general solutions to the diophantine without heavy computations.

3. The Second Form: $3A^4 + B^4 = N^2$

So we apply the same logic to deduce $3C^4 + 1 = N^2$.

Here, note that there are easy cases that can be found such as $C = 0, 1, 2$. Since $\alpha = 3$ and $\beta = 1$, we get

$$4+8x+24x^2+8x^3+4x^4 = N^2, 1+2x+6x^2+2x^3+x^4 = N^2, \text{ then } (1+x+x^2)^2+3x^2 = N^2$$

So following the same steps we get $x = 2pq$ where:

$$x = \frac{-4}{7}, \frac{231}{448}, \text{ so } C = \frac{-3}{11}$$

.

$$\sqrt{3C^4+1} = \frac{122}{121}$$

. And we are done.

4. Bonus Enrichment: Euler's Conjecture on Diophantines

This was an interesting conjecture proposed by Euler that any diophantine of $a^4 = b^4+c^4+d^4$ would be insoluble. Elkies disproves this fact using the parameterization of surface $r^4 + s^4 + t^4 = 1$ and elliptic curves to find the existence of a rational point. One counterexample is:

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4.$$