Unravelling Euler's ingenious Integration: A Comprehensive Analysis of E695.

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Abstract

In this paper, we will provide an exposition of Euler's 1797 paper, "Integratio Succincta Formulae Integralis Maxime Memorabilis" (E695), in which Euler calculated a titular integral using a series of intricate substitutions and innovative ideas.

1 Introduction

In this section, we will firstly discuss about several results[1] which were commonly used in Euler's day, and will also be used further along the paper as we solve Euler's integral formula. Eventually, we will define the integral formula that Euler solved in his paper[2] [Eul97], and define the substitutions Euler used in solving the integral. By the end of this section, we hope to get an idea of how Euler approached to solve the proposed integral formula.

Theorem 1.1.

$$\arctan(x) - \arctan(y) = \arctan\left(\frac{x-y}{1+xy}\right)$$

if $x > 0$, $y > 0$, and $xy < 1$.

Proof. Let $\alpha = \arctan(x)$ and $\beta = \arctan(y)$. Then, clearly $x = \tan\alpha$ and $y = \tan\beta$. Now,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$\implies \tan(\alpha - \beta) = \frac{x-y}{1+xy}$$

Hence,

$$\alpha - \beta = \arctan\frac{x-y}{1+xy}$$

$$\implies \arctan x - \arctan y = \arctan\frac{x-y}{1+xy}.$$

[1] Note that these results were proved in Euler's paper, and were assumed to be already known by the reader. We will still add these though, to make the exposition much easier to understand.

[2] Although we cite the original text of Euler's paper, we have followed through the translated version of the paper as well, and it can be found at http://eulerarchive.maa.org/docs/translations/E695en.pdf

We will still consider the limits $x > 0$, $y > 0$, and $xy > 1$ so that the equation is defined in $\mathbb{R}$. ■

Theorem 1.2.

$$\int \frac{dp}{1-p^3} = \frac{1}{3} \ln \frac{\sqrt{1+p+p^2}}{1-p} + \frac{1}{\sqrt{3}} \arctan \frac{p\sqrt{3}}{2+p}.$$

Proof. Using partial fractions, we can evaluate the integral as

$$\int \frac{dp}{1-p^3} = \int \left(\frac{p+2}{3(p^2+p+1)} - \frac{1}{3(p-1)} \right) dp.$$

Solving this integral, we get that

$$\begin{aligned} \int \left(\frac{p+2}{3(p^2+p+1)} - \frac{1}{3(p-1)} \right) dp &= \frac{1}{3} \int \frac{p+2}{3(p^2+p+1)} dp - \frac{1}{3} \int \frac{1}{3(p-1)} dp \\ &= \frac{1}{3} \ln \frac{\sqrt{1+p+p^2}}{1-p} + \frac{1}{\sqrt{3}} \arctan \frac{p\sqrt{3}}{2+p}. \end{aligned}$$

■

Theorem 1.3. *For some value of* $x \in \mathbb{R}$*,*

$$1 + x + x^2 = \frac{1-x^3}{1-x}.$$

Proof. The proof of this theorem is very simple. We can see that $(1-x^3)$ can be factored in the form

$$(1-x^3) = (1-x)(1+x+x^2).$$

One can easily check this through the Polynomial Long Division Method, and hence, we obtain the desired equation. ■

Theorem 1.4.

$$\arctan(t\sqrt{-1}) = \int \frac{\sqrt{-1}}{1-t^2} dt = \frac{\sqrt{-1}}{2} \ln \frac{1+t}{1-t}.$$

Proof. We already know that

$$\int \frac{dx}{1-x^2} = \arctan x.$$

So it is clear that

$$\arctan(t\sqrt{-1}) = \int \frac{\sqrt{-1}}{1-t^2} dt.$$

Now, we will try to solve this integral using partial fractions.

$$\begin{aligned} \int \frac{\sqrt{-1}}{1-t^2} dt &= \sqrt{-1} \int \left(\frac{1}{1+t} \right) \left(\frac{1}{1-t} \right) dt \\ &= \sqrt{-1} \left(\int \frac{1}{2(1+t)} + \frac{1}{2(1-t)} dt \right) \\ &= \frac{\sqrt{-1}}{2} \left(\ln|1+t| - \ln|1-t| \right) \\ &= \frac{\sqrt{-1}}{2} \ln \frac{1+t}{1-t}. \end{aligned}$$

$\blacksquare$

In [Eul97], Euler focuses on solving the integral of the form

$$\int \frac{dz}{(3 \pm z^2)\sqrt[3]{1 \pm 3z^2}}. \tag{1.1}$$

In doing so for the positive case first, he uses the substitution $v = \sqrt[3]{1+3z^2}$, and then considers $p = \frac{1+z}{v}$ and $q = \frac{1-z}{v}$. With the help of Theorem 1.3 and through clever manipulation of these substitutions, Euler obtains a generalized differential formula of the form discussed in Theorem 1.2. Then, he further evaluates the integral using Theorem 1.1 and after several other substitutions, he finally evaluate the integral. For the negative case, he uses the same solution as he considers a substitution to represent the integral in the form of the positive case of (1.1). Then, using Theorem 1.4, and several other substitutions, he further evaluates the negative case of the proposed integral, eventually eliminating the imaginary numbers.

2 The most Memorable Integral Formula

In Euler's words, (1.1) was considered as "The most Memorable Integral Formula", and in this section, we will try understand how that actually makes sense. We will first solve this integral through substitutions , and develop a sense of the ultimate solution throughout this section.

2.1 The Positive Case

We will first consider the integral with the positive signs, and let

$$dV = \frac{dz}{(3+z^2)\sqrt[3]{1+3z^2}}. \tag{2.1}$$

Using the substitution $v^3 = 1 + 3z^2$, we obtain that

$$\begin{aligned} 3v^2 dv &= 6z dz \\ dz &= \frac{v^2}{2z} dv. \end{aligned}$$

Hence, we can rewrite (2.1) as

$$dV = \frac{v}{2z(3+z^2)} dv. \tag{2.2}$$

Now, let $p = \frac{1+z}{v}$ and $q = \frac{1-z}{v}$ such that

$$\begin{aligned}
p^3 + q^3 &= \frac{2+6z^2}{v^3} = 2, \\
p^3 - q^3 &= \frac{2z^3+6z}{v^3} = \frac{2z(3+z^2)}{v^3}, \\
p + q &= \frac{2}{v}, \\
p - q &= \frac{2z}{v}, \\
pq &= \frac{1-z^2}{v^2}, \text{ and} \\
v(dp+dq) + dv(p+q) = 0 &\implies dp + dq = -\frac{2}{v^2}dv.
\end{aligned} \tag{2.3}$$

Rewriting (2.2),

$$\begin{aligned}
dV &= -\frac{v^3(dp+dq)}{2v^3(p^3-q^3)} \\
dV &= -\frac{dp+dq}{2(p^3-q^3)} \\
&= -\frac{dp+dq}{(p^3+q^3)(p^3-q^3)}.
\end{aligned}$$

Thus,

$$dV = -\frac{dp+dq}{2(p^3-q^3)}.$$

We will now try to split this formula into two parts and use Theorem 1.2 to evaluate this equation. Let

$$dP = \frac{dp}{p^3-q^3}, \quad \text{and} \quad dQ = \frac{dq}{p^3-q^3}.$$

Then,

$$dV = -\frac{1}{2}dP - \frac{1}{2}dQ. \tag{2.4}$$

From (2.3), since $p^3 + q^3 = 2$, we can observe that

$$dP = -\frac{dp}{2(1-q^3)}, \quad \text{and} \quad dQ = \frac{dq}{2(1-q^3)}.$$

Hence, we can rewrite (2.4) as

$$4dV = \frac{dp}{1-p^3} - \frac{dq}{1-q^3}.$$

Using Theorem 1.2, we can evaluate this differential equation as

$$4V = \int \frac{dp}{1-p^3} - \int \frac{dq}{1-q^3}$$

$$
\begin{aligned}
4V = &\left(\frac{1}{3}\ln\frac{\sqrt{1+p+p^2}}{1-p} + \frac{1}{\sqrt{3}}\arctan\frac{p\sqrt{3}}{2+p}\right) \\
&- \left(\frac{1}{3}\ln\frac{\sqrt{1+q+q^2}}{1-q} + \frac{1}{\sqrt{3}}\arctan\frac{q\sqrt{3}}{2+q}\right)
\end{aligned} \tag{2.5}
$$

Using Theorem 1.3, we can observe that $1+p+p^2 = \frac{1-p^3}{1-p}$, and thus,

$$
\begin{aligned}
4V = &\left(\frac{1}{6}\ln\frac{1-p^3}{(1-p)^3} + \frac{1}{\sqrt{3}}\arctan\frac{p\sqrt{3}}{2+p}\right) \\
&- \left(\frac{1}{3}\ln\frac{1-q^3}{(1-q)^3} + \frac{1}{\sqrt{3}}\arctan\frac{q\sqrt{3}}{2+q}\right)
\end{aligned}
$$

$$
\implies 4V = \frac{1}{6}\left(\ln\frac{1-p^3}{(1-p)^3} - \ln\frac{1-q^3}{(1-q)^3}\right) + \frac{1}{\sqrt{3}}\left(\arctan\frac{p\sqrt{3}}{2+p} - \arctan\frac{q\sqrt{3}}{2+q}\right).
$$

Through logarithmic manipulation, and Theorem 1.1, we can see that

$$
\ln\frac{1-p^3}{(1-p)^3} - \ln\frac{1-q^3}{(1-q)^3} = \ln\frac{1-p^3}{1-q^3} - \ln\frac{(1-p)^3}{(1-q)^3}, \quad \text{and}
$$

$$
\arctan\frac{p\sqrt{3}}{2+p} - \arctan\frac{q\sqrt{3}}{2+q} = \arctan\frac{(p-q)\sqrt{3}}{2+p+q+2pq}.
$$

Furthermore, we also know that $1-p^3 = q^3-1 = -(1-q^3)$, so we would get the term $\ln\sqrt{-1}$, which Euler treats as a constant in [Eul97], and omits the indeterminate complex quantity, thus obtaining

$$
\begin{aligned}
4V &= \frac{1}{6}\ln\frac{(1-p)^3}{(1-q)^3} + \frac{1}{\sqrt{3}}\arctan\frac{(p-q)\sqrt{3}}{2+p+q+2pq} \\
4V &= \frac{1}{2}\ln\frac{1-p}{1-q} + \frac{1}{\sqrt{3}}\arctan\frac{(p-q)\sqrt{3}}{2+p+q+2pq}.
\end{aligned}
$$

Retracing this equation back to v and z using $p = \frac{1+z}{v}$ and $q = \frac{1-z}{v}$, we get that

$$
\frac{1}{2}\ln\frac{1-p}{1-q} = \frac{1}{2}\ln\frac{v-1-z}{v-1+z} = \frac{1}{2}\ln\frac{1-v-z}{1-v+z}, \quad \text{and}
$$

$$
\frac{1}{\sqrt{3}}\arctan\frac{(p-q)\sqrt{3}}{2+p+q+2pq} = \frac{1}{\sqrt{3}}\arctan\frac{\left(\frac{2z}{v}\right)\sqrt{3}}{2+\frac{2}{v}+2\left(\frac{1-z^2}{v^2}\right)} = \frac{1}{\sqrt{3}}\arctan\frac{vz\sqrt{3}}{1+v+v^2-z^2}.
$$

Therefore,

$$4V = \frac{1}{2}\ln\frac{1-v-z}{1-v+z} + \frac{1}{\sqrt{3}}\arctan\frac{vz\sqrt{3}}{1+v+v^2-z^2}$$

$$\implies V = \frac{1}{8}\ln\frac{1-v-z}{1-v+z} + \frac{1}{4\sqrt{3}}\arctan\frac{vz\sqrt{3}}{1+v+v^2-z^2}$$

where v is equal to $\sqrt[3]{1+3z^2}$.

2.2 The Negative Case

We will now solve the integral with the negative signs, and let $z = y\sqrt{-1}$ in (2.1), which gives us that

$$dV = \frac{dy\sqrt{-1}}{(3-y^2)\sqrt[3]{1-3y^2}}. \tag{2.6}$$

When we consider z as $y\sqrt{-1}$, we get an integral which is of the same form as the one we just solved in §2.1, so we can show that (2.6) is just equal to

$$\frac{1}{8}\ln\frac{1-v-y\sqrt{-1}}{1-v+y\sqrt{-1}} + \frac{1}{4\sqrt{3}}\arctan\frac{vy\sqrt{3}\sqrt{-1}}{1+v+v^2+y^2}, \tag{2.7}$$

where $v = \sqrt[3]{1-3y^2}$.

However, we can further evaluate (2.6) through Theorem 1.4 by considering t as $\frac{vy\sqrt{3}}{1+v+v^2+y^2}$. Then,

$$\arctan\left(t\sqrt{-1}\right) = \frac{\sqrt{-1}}{3}\ln\frac{1+v+v^2+y^2+vy\sqrt{3}}{1+v+v^2+y^2-vy\sqrt{3}}. \tag{2.8}$$

By considering $t = u\sqrt{-1}$ and through Theorem 1.4, we will get that

$$-\arctan u = \frac{\sqrt{-1}}{2}\ln\frac{1+u\sqrt{-1}}{1-u\sqrt{-1}}$$

$$\implies 2\sqrt{-1}\arctan u = \ln\frac{1+u\sqrt{-1}}{1-u\sqrt{-1}}.$$

Now, we know that $t = u\sqrt{-1}$ and for our case, when we compare the above equation with the first term on the right hand side in (2.7), we can see that $u = -\frac{y}{1-v}$. Hence,

$$2\sqrt{-1}\arctan\left(-\frac{y}{1-v}\right) = \ln\frac{1-v-y\sqrt{-1}}{1-v+y\sqrt{-1}}. \tag{2.9}$$

With these values [(2.8) and (2.9)], we can rewrite the solution for the integral, (2.7) as

$$\int\frac{dy\sqrt{-1}}{(3-y^2)\sqrt[3]{1-3y^2}} = -\frac{\sqrt{-1}}{4}\arctan\frac{y}{1-v} + \frac{\sqrt{-1}}{8\sqrt{3}}\ln\frac{1+v+v^2+y^2+vy\sqrt{3}}{1+v+v^2+y^2-vy\sqrt{3}}$$

Here, we divide both side by $\sqrt{-1}$ to eliminate the imaginary numbers so that we can obtain the integration

$$\int \frac{dy}{(3-y^2)\sqrt[3]{1-3y^2}} = \frac{1}{8\sqrt{3}} \ln \frac{1+v+v^2+y^2+vy\sqrt{3}}{1+v+v^2+y^2-vy\sqrt{3}} - \frac{1}{4} \arctan \frac{y}{1-v}$$

Since $v = \sqrt[3]{1-3y^2}$, $3y^2 = 1 - v^3$ and multiplying the fraction joined to the logarithm above and below by $(1-v)$, the fraction would be equal to

$$\frac{(1-v)(1+v+v^2+y^2+vy\sqrt{3})}{(1-v)(1+v+v^2+y^2-vy\sqrt{3})} = \frac{y(4-v)+v(1-v)\sqrt{3}}{y(4-v)-v(1-v)\sqrt{3}}.$$

Therefore, the integral takes the form

$$\int \frac{dy}{(3-y^2)\sqrt[3]{1-3y^2}} = \frac{1}{8\sqrt{3}} \ln \frac{y(4-v)+v(1-v)\sqrt{3}}{y(4-v)-v(1-v)\sqrt{3}} - \frac{1}{4} \arctan \frac{y}{1-v},$$

where $v = \sqrt[3]{1-3y^2}$.

This was essentially the method Euler used to solve the proposed integral formula, (1.1). However, Euler also mentioned about another approach in [Eul97], where he obtains a differential formula through the proposed integral formula. We will look at that in the next section.

3 A succinct Differential Formula

Although we have beautifully shown how the integral is solved above, we will explore another approach to the solution in this formula using the substitution $z = \frac{1+x}{1-x}$. We will then use logarithmic differentiation to produce a differential equation, whose solution we won't evaluate further, but should be the same as the once discussed above. This is supposedly a much easier and a comprehended approach to solve the proposed formula by Euler.

Let $z = \frac{1+x}{1-x}$. Then,

$$\begin{aligned} dz &= \frac{dx(1-x)+(1+x)dx}{(1-x)^2} = \frac{2dx}{(1-x)^2}, \\ 3+z^2 &= \frac{4-4x+4x^2}{(1-x)^2} = \frac{4(1+x^3)}{(1+x)(1-x)^2}, \\ 1+3z^2 &= \frac{4(1-x^3)}{(1-x)3} \implies \sqrt[3]{1+3z^2} = \frac{\sqrt[3]{4(1-x^3)}}{1-x}. \end{aligned}$$

Using these substitutions, we can rewrite (2.1) as

$$dV = \frac{1}{2\sqrt[3]{4}} \cdot \frac{(1-x^2)}{(1+x^3)\sqrt[3]{1-x^3}} dx.$$

Using partial fractions, we could represent the integration of the above formula as

$$2\sqrt[3]{4}V = \int \frac{dx}{(1+x^3)\sqrt[3]{1-x^3}} - \int \frac{x^2dx}{(1+x^3)\sqrt[3]{1-x^3}}. \tag{3.1}$$

Let $t = \frac{x}{\sqrt[3]{1-x^3}}$. Then, $x^3 = \frac{t^3}{1+t^3}$ and when we try to differentiate x^3 using logarithmic differentiation,

$$\begin{aligned}
\ln x^3 &= \ln \frac{t^3}{1+t^3} \\
\frac{1}{x^3}(3x^2)\frac{dx}{dt} &= \left(\frac{1+t^3}{t^3}\right)\left(\frac{3t^2(1+t^3)-t^3(3t^2)}{(1+t^3)^2}\right) \\
\frac{3}{x}\frac{dx}{dt} &= \frac{3}{t(1+t^3)} \\
\implies \frac{dx}{x} &= \frac{dt}{1(1+t^3)}.
\end{aligned}$$

For the later part of the integral, we will make a different substitution. Let $u = \sqrt[3]{1-x^3}$, which then can be interpreted as $x^3 = 1-u^3$. Then, $1+x^3 = 2-u^3$ and through implicit differentiation, we obtain that $x^2dx = -u^2du$. Using these equations we can rewrite the second part of the integral in (3.1) as

$$\int \frac{x^2dx}{(1+x^3)\sqrt[3]{1-x^3}} = -\int \frac{udu}{2-u^3}.$$

Thus, (3.1) can be written as

$$2V\sqrt[3]{4} = \int \frac{dt}{1+2t^3} + \int \frac{udu}{2-u^3}. \tag{3.2}$$

In this way, Euler transforms the proposed formula into two other succinct formulas, through which, Euler realized that the proposed formula could be solved through understood rules. However, it is easily visible that the final formula in (3.2) is much more easily obtained by our method in §2 than if we wished to evaluate (3.2) further. This was the main reason Euler further reinstated the title of "The Most Memorable Integral Formula" mostly to the other method of the solution, which he believed to snatch away the victory as the solution for the proposed formula.

Furthermore, when we wish to handle the other case of the formula

$$\frac{dz}{(3-z^2)\sqrt[3]{1-3z^2}}$$

in the same manner, we would want to substitute $z = \frac{1+x}{1-x}\sqrt{-1}$, and the integration could not be presented unless by proceeding by the means of imaginary numbers. This was where Euler ended his paper [Eul97], as he urged for the use of the Calculus of Imaginary numbers in integral calculus. Eventually, Euler discussed about this integral formula once again in his paper 1801 paper[3] titled "De insigni usu calculi imaginariorum in calculo integrali" [Eul01], where he talks about several other integral formulas as well, which he solves thoroughly solves through the use of imaginary numbers and methods from complex analysis.

References

[Eul97] Leonhard Euler. Integratio succincta formulae integralis maxime memorabilis z/((3±zz)(1±3zz) 1/3). *Nova Acta Academiae Scientiarum Imperialis Petropolitanae*, pages 20–26, 1797.

[Eul01] Leonhard Euler. De insigni usu calculi imaginariorum in calculo integrali. *Nova Acta Academiae Scientiarum Imperialis Petropolitanae*, pages 3–21, 1801.

[3]Note that this paper does not have a translation version in English, yet.