Summation of the series $\sin(\phi)^\lambda + \sin(2\phi)^\lambda + ... \sin(n\phi)^\lambda$ $\cos(\phi)^\lambda + \cos(2\phi)^\lambda + ... \cos(n\phi)^\lambda$

Julian Stockton

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Abstract

In this paper, I will give a sampling of some of Euler's techniques for summing the series $\sin(\phi)^\lambda + \sin(2\phi)^\lambda + ... \sin(n\phi)^\lambda$ and $\cos(\phi)^\lambda + \cos(2\phi)^\lambda + ... \cos(n\phi)^\lambda$. This material is mostly based on **[E447]**.

1 Expressions to aid in summation

Assign p and q to be the following:

$$p = \cos(\phi) + i\sin(\phi) = \frac{1}{2}(e^{\phi i} + e^{-\phi i}) + i\frac{1}{2i}(e^{\phi i} - e^{-\phi i}) = e^{\phi i}$$

$$q = \cos(\phi) - i\sin(\phi) = \frac{1}{2}(e^{\phi i} + e^{-\phi i}) - i\frac{1}{2i}(e^{\phi i} - e^{-\phi i}) = e^{-\phi i}$$

From these definitions:

$$\cos(n\phi) = \frac{1}{2}(e^{n\phi i} + e^{-n\phi i}) = \frac{1}{2}(p^n + q^n) \qquad (1.1)$$

$$\sin(n\phi) = \frac{1}{2i}(e^{n\phi i} - e^{-n\phi i}) = \frac{1}{2i}(p^n - q^n) \qquad (1.2)$$

$$pq = e^{\phi i - \phi i} = 1 \qquad (1.3)$$

Summing the powers of p^α and q^α then yields (as they are both geometric series):

$$\sum_{j=1}^{n} p^{j\alpha} = \frac{p^\alpha p^{n\alpha} - p^\alpha}{p^\alpha - 1} = \frac{p^\alpha - p^{(n+1)\alpha}}{1 - p^\alpha}$$

$$\sum_{j=1}^{n} q^{j\alpha} = \frac{q^\alpha q^{n\alpha} - q^\alpha}{q^\alpha - 1} = \frac{q^\alpha - q^{(n+1)\alpha}}{1 - q^\alpha}$$

Theorem 1. $\sum_{j=1}^{n}(p^{j\alpha} + q^{j\alpha}) = \frac{\cos(n\alpha\phi) - \cos((n+1)\alpha\phi)}{1 - \cos(\alpha\phi)} - 1$

Proof.

$$\sum_{j=1}^{n} p^{j\alpha} + \sum_{j=1}^{n} q^{j\alpha} = \frac{p^{\alpha} - p^{(n+1)\alpha}}{1 - p^{\alpha}} + \frac{q^{\alpha} - q^{(n+1)\alpha}}{1 - q^{\alpha}} =$$

$$\frac{p^{\alpha} - p^{(n+1)\alpha} - p^{\alpha}q^{\alpha} + p^{(n+1)\alpha}q^{\alpha} + q^{\alpha} - q^{(n+1)\alpha} - p^{\alpha}q^{\alpha} + p^{\alpha}q^{(n+1)\alpha}}{1 - p^{\alpha} - q^{\alpha} + p^{\alpha}q^{\alpha}} =$$

$$\frac{p^{\alpha} - p^{(n+1)\alpha} - 1^{\alpha} + p^{n\alpha}1^{\alpha} + q^{\alpha} - q^{(n+1)\alpha} - 1^{\alpha} + 1^{\alpha}q^{n\alpha}}{1 - p^{\alpha} - q^{\alpha} + 1^{\alpha}} = \qquad \text{(by 1.3)}$$

$$\frac{p^{\alpha} + q^{\alpha} - p^{(n+1)\alpha} - q^{(n+1)\alpha} - 2 + p^{n\alpha} + q^{n\alpha}}{2 - p^{\alpha} - q^{\alpha}} =$$

$$\frac{2\cos(\alpha\phi) - 2\cos((n+1)\alpha\phi) - 2 + 2\cos(n\alpha\phi)}{2 - 2\cos(\alpha\phi)} = \frac{\cos(n\alpha\phi) - \cos((n+1)\alpha\phi)}{1 - \cos(\alpha\phi)} - 1$$

The above expression was obtained by multiplying both sides of 1.1 by 2 and plugging in α, $(n+1)\alpha$, and $n\alpha$ for n. □

Theorem 2. $\sum_{j=1}^{n}(p^{j\alpha} - q^{j\alpha}) = \frac{\sin(\alpha\phi) - \sin((n+1)\alpha\phi) + \sin(n\alpha\phi)}{1 - \cos(\alpha\phi)} i$

Proof.

$$\sum_{j=1}^{n} p^{j\alpha} - \sum_{j=1}^{n} q^{j\alpha} = \frac{p^{\alpha} - p^{(n+1)\alpha}}{1 - p^{\alpha}} + \frac{-q^{\alpha} + q^{(n+1)\alpha}}{1 - q^{\alpha}} =$$

$$\frac{p^{\alpha} - p^{(n+1)\alpha} - p^{\alpha}q^{\alpha} + p^{(n+1)\alpha}q^{\alpha} - q^{\alpha} + q^{(n+1)\alpha} + p^{\alpha}q^{\alpha} - p^{\alpha}q^{(n+1)\alpha}}{1 - p^{\alpha} - q^{\alpha} + p^{\alpha}q^{\alpha}} =$$

$$\frac{p^{\alpha} - p^{(n+1)\alpha} - 1^{\alpha} + p^{n\alpha}1^{\alpha} - q^{\alpha} + q^{(n+1)\alpha} + 1^{\alpha} - 1^{\alpha}q^{n\alpha}}{1 - p^{\alpha} - q^{\alpha} + 1^{\alpha}} = \qquad \text{(by 1.3)}$$

$$\frac{p^{\alpha} - q^{\alpha} - p^{(n+1)\alpha} + q^{(n+1)\alpha} - 1 + 1 + p^{n\alpha} - q^{n\alpha}}{2 - p^{\alpha} - q^{\alpha}} =$$

$$\frac{2i\sin(\alpha\phi) - 2i\sin((n+1)\alpha\phi) + 2i\sin(n\alpha\phi)}{2 - 2\cos(\alpha\phi)} =$$

$$\frac{\sin(\alpha\phi) - \sin((n+1)\alpha\phi) + \sin(n\alpha\phi)}{1 - \cos(\alpha\phi)} i$$

The above expression was obtained by multiplying both sides of 1.2 by $2i$ and plugging in α, $(n+1)\alpha$, and $n\alpha$ for n, and by multiplying both sides of 1.1 by 2 and plugging in α for n. □

2 Example 1: $\lambda = 7$

The two series we are summing are $\sum_{j=1}^{n}(\sin(j\phi))^7$ and $\sum_{j=1}^{n}(\cos(j\phi))^7$.

$$\sum_{j=1}^{n}(\sin(j\phi))^7 = \sum_{j=1}^{n}\frac{1}{(2i)^7}(p^j - q^j)^7 =$$

$$\frac{i}{128}\sum_{j=1}^{n}(p^{7j} - 7p^{6j}q^j + 21p^{5j}q^{2j} - 35p^{4j}q^{3j} + 35p^{3j}q^{4j} - 21p^{2j}q^{5j} + 7p^jq^{6j} - q^{7j}) =$$

$$\frac{i}{128}(\sum_{j=1}^{n}(p^{7j} - q^{7j}) - 7\sum_{j=1}^{n}(p^{5j}1^j - 1^jq^{5j}) +$$

$$21\sum_{j=1}^{n}(p^{3j}1^{2j} - 1^{2j}p^{3j}) - 35\sum_{j=1}^{n}(p^j1^{3j} - 1^{3j}q^j)) =$$

$$-\frac{\sin(7\phi) - \sin(7(n+1)\phi) + \sin(7n\phi)}{128(1-\cos(7\phi))} + 7\frac{\sin(5\phi) - \sin(5(n+1)\phi) + \sin(5n\phi)}{128(1-\cos(5\phi))} -$$

$$21\frac{\sin(3\phi) - \sin(3(n+1)\phi) + \sin(3n\phi)}{128(1-\cos(3\phi))} + 35\frac{\sin(\phi) - \sin((n+1)\phi) + \sin(n\phi)}{128(1-\cos(\phi))}$$

Note that the initial sum of sines was transformed into a sum of powers of ps and qs by application of 1.2. That sum was then expanded out and simplified by 1.3. The resulting 4 sums were then converted into fractions of sines and cosines using Theorem 2.

$$\sum_{j=1}^{n}(\cos(j\phi))^7 = \sum_{j=1}^{n}\frac{1}{2^7}(p^j + q^j)^7 =$$

$$\frac{1}{128}\sum_{j=1}^{n}(p^{7j} + 7p^{6j}q^j + 21p^{5j}q^{2j} + 35p^{4j}q^{3j} + 35p^{3j}q^{4j} + 21p^{2j}q^{5j} + 7p^jq^{6j} + q^{7j}) =$$

$$\frac{1}{128}(\sum_{j=1}^{n}(p^{7j} + q^{7j}) + 7\sum_{j=1}^{n}(p^{5j}1^j + 1^jq^{5j}) +$$

$$21\sum_{j=1}^{n}(p^{3j}1^{2j} + 1^{2j}p^{3j}) + 35\sum_{j=1}^{n}(p^j1^{3j} + 1^{3j}q^j)) =$$

$$\frac{\cos(7n\phi) - \cos(7(n+1)\phi)}{128(1-\cos(7\phi))} + 7\frac{\cos(5n\phi) - \cos(5(n+1)\phi)}{128(1-\cos(5\phi))} +$$

$$21\frac{\cos(3n\phi) - \cos(3(n+1)\phi)}{128(1-\cos(3\phi))} + 35\frac{\cos(n\phi) - \cos((n+1)\phi)}{128(1-\cos(\phi))} - \frac{1}{32}$$

Note that the initial sum of cosines was transformed into a sum of powers of ps and qs by application of 1.1. That sum was then expanded out and simplified by 1.3. The resulting 4 sums were then converted into fractions of cosines using Theorem 1.

3 Example 2: $\lambda = 11$

The two series we are summing are $\sum_{j=1}^{n}(\sin(j\phi))^{11}$ and $\sum_{j=1}^{n}(\cos(j\phi))^{11}$.

$$\sum_{j=1}^{n}(\sin(j\phi))^{11} = \sum_{j=1}^{n}\frac{1}{(2i)^{11}}(p^j - q^j)^{11} =$$

$$\frac{i}{2048}\sum_{j=1}^{n}(p^{11j} - 11p^{10j}q^j + 55p^{9j}q^{2j} - 165p^{8j}q^{3j} + 330p^{7j}q^{4j} - 462p^{6j}q^{5j} +$$

$$462p^{5j}q^{6j} - 330p^{4j}q^{7j} + 165p^{3j}q^{8j} - 55p^{2j}q^{9j} + 11p^jq^{10j} - q^{11j}) =$$

$$\frac{i}{2048}(\sum_{j=1}^{n}(p^{11j} - q^{11j}) - 11\sum_{j=1}^{n}(p^{9j}1^j - 1^jq^{9j}) + 55\sum_{j=1}^{n}(p^{7j}1^{2j} - 1^{2j}q^{7j}) -$$

$$165\sum_{j=1}^{n}(p^{5j}1^{3j} - 1^{3j}q^{5j}) + 330\sum_{j=1}^{n}(p^{3j}1^{4j} - 1^{4j}q^{3j}) - 462\sum_{j=1}^{n}(p^j1^{5j} - 1^{5j}q^j)) =$$

$$-\frac{\sin(11\phi) - \sin(11(n+1)\phi) + \sin(11n\phi)}{2048(1-\cos(11\phi))} + 11\frac{\sin(9\phi) - \sin(9(n+1)\phi) + \sin(9n\phi)}{2048(1-\cos(9\phi))} -$$

$$55\frac{\sin(7\phi) - \sin(7(n+1)\phi) + \sin(7n\phi)}{2048(1-\cos(7\phi))} + 165\frac{\sin(5\phi) - \sin(5(n+1)\phi) + \sin(5n\phi)}{2048(1-\cos(5\phi))} -$$

$$330\frac{\sin(3\phi) - \sin(3(n+1)\phi) + \sin(3n\phi)}{2048(1-\cos(3\phi))} + 462\frac{\sin(\phi) - \sin((n+1)\phi) + \sin(n\phi)}{2048(1-\cos(\phi))}$$

Note that the initial sum of sines was transformed into a sum of powers of ps and qs by application of 1.2. That sum was then expanded out and simplified by 1.3. The resulting 6 sums were then converted into fractions of sines and cosines using Theorem 2.

$$\sum_{j=1}^{n}(\cos(j\phi))^{11} = \sum_{j=1}^{n}\frac{1}{2^{11}}(p^j + q^j)^{11}$$

$$\frac{1}{2048}\sum_{j=1}^{n}(p^{11j} + 11p^{10j}q^j + 55p^{9j}q^{2j} + 165p^{8j}q^{3j} + 330p^{7j}q^{4j} + 462p^{6j}q^{5j} + 462p^{5j}q^{6j} +$$

$$330p^{4j}q^{7j} + 165p^{3j}q^{8j} + 55p^{2j}q^{9j} + 11p^jq^{10j} + q^{11j}) =$$

$$\frac{1}{2048}(\sum_{j=1}^{n}(p^{11j} + q^{11j}) + 11\sum_{j=1}^{n}(p^{9j}1^j + 1^jq^{9j}) + 55\sum_{j=1}^{n}(p^{7j}1^{2j} + 1^{2j}q^{7j}) +$$

$$165\sum_{j=1}^{n}(p^{5j}1^{3j} + 1^{3j}q^{5j}) + 330\sum_{j=1}^{n}(p^{3j}1^{4j} + 1^{4j}q^{3j}) + 462\sum_{j=1}^{n}(p^j1^{5j} + 1^{5j}q^j)) =$$

$$\frac{\cos(11n\phi) - \cos(11(n+1)\phi)}{2048(1-\cos(11\phi))} + 11\frac{\cos(9n\phi) - \cos(9(n+1)\phi)}{2048(1-\cos(9\phi))} +$$

$$55\frac{\cos(7n\phi)-\cos(7(n+1)\phi)}{2048(1-\cos(7\phi))}+165\frac{\cos(5n\phi)-\cos(5(n+1)\phi)}{2048(1-\cos(5\phi))}+$$

$$330\frac{\cos(3n\phi)-\cos(3(n+1)\phi)}{2048(1-\cos(3\phi))}+462\frac{\cos(n\phi)-\cos((n+1)\phi)}{2048(1-\cos(\phi))}-\frac{3}{1024}$$

Note that the initial sum of cosines was transformed into a sum of powers of ps and qs by application of 1.1. That sum was then expanded out and simplified by 1.3. The resulting 6 sums were then converted into fractions of cosines using Theorem 1.

4 Infinite versions of the series

Note that for even λ, as neither $\lim_{x\to\infty}\sin(x\phi)^\lambda$ nor $\lim_{x\to\infty}\cos(x\phi)^\lambda$ are defined (this is technically also true for odd λ, but in that case the sum diverges), and as $0\leq\sin(x\phi)^\lambda$, for even λ $\sum_{j=1}^{\infty}\sin(j\phi)$ and $\sum_{j=1}^{\infty}\cos(j\phi)$ can both be said to be ∞. However, taking $\lambda=7$ as an example gives (omitting terms having j in them because they are undefined, but still between -1 and 1):

$$\sum_{j=1}^{\infty}\sin(j\phi)^7=-\frac{\sin(7\phi)}{128(1-\cos(7\phi))}+\frac{7\sin(5\phi)}{128(1-\cos(5\phi))}-\frac{21\sin(3\phi)}{128(1-\cos(3\phi))}+\frac{35\sin(\phi)}{128(1-\cos(\phi))}$$

$$\sum_{j=1}^{\infty}\cos(j\phi)^7=\frac{-1}{32}$$

And for $\lambda=11$:

$$\sum_{j=1}^{\infty}\sin(j\phi)^{11}=-\frac{\sin(11\phi)}{2048(1-\cos(11\phi))}+\frac{11\sin(9\phi)}{2048(1-\cos(9\phi))}-\frac{55\sin(7\phi)}{2048(1-\cos(7\phi))}+$$

$$\frac{165\sin(5\phi)}{2048(1-\cos(5\phi))}-\frac{330\sin(3\phi)}{2048(1-\cos(3\phi))}+\frac{462\sin(\phi)}{2048(1-\cos(\phi))}$$

$$\sum_{j=1}^{\infty}\cos(j\phi)^{11}=\frac{-3}{1024}$$

Note that the above equals signs are not rigorous, as they are more similar to the equals sign in the statement $\sum_{j=0}^{\infty}2^j=-1$.