Observations on Continued Fractions

Inho Ryu

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Abstract

This paper covers Leonhard Euler's observations on continued fractions. This includes Euler's definition of a continued fraction, the infinite series that can be obtained from the definition, and the properties of variations of the continued fraction, mainly of $\int \frac{dx}{1+x}$.

1 Introduction

While there were a few cases appearing in the sixteenth and seventeeth century, most of the elementary theory of continued fractions were developed by Leonhard Euler in a single paper written in 1737 called "De fractionibus continuis dissertatio" ("Observations on continued fractions") [EA18]. In his paper, Euler presented continued fractions as an alternative to infinite series or products for representing different types of quantities. He also described basic properties of continued fractions and also some special cases of them, which I will cover in this paper.

2 Main Results

Euler presents a continued fraction as in the form

$$A + \cfrac{B}{C + \cfrac{D}{E + \cfrac{F}{G + \cfrac{H}{I + \text{etc.}}}}}$$

whose value can be found by continuing the following series to infinity

$$A + \frac{B}{1P} - \frac{BD}{PQ} + \frac{BDF}{QR} - \frac{BDFH}{RS} - \text{etc.},$$

where

$$P = C, Q = EP + D, R = GQ + FP, S = IR + HQ, \text{etc.}$$

This series always converges, no matter how B, C, D, E, F, etc. either grow or decrease, just as long as they are positive. This is because any arbitrary term is smaller than the preceding one, but greater than the following as immediately seen by the rule used to form the values P, Q, R, S etc.

Now if we consider the infinite series

$$\frac{B}{P} - \frac{BD}{PQ} + \frac{BDF}{QR} - \frac{BDFH}{RS} + \text{etc.},$$

then its sum can be represented by the following continued fraction. Since

$$C = P, E = \frac{Q-D}{P}, G = \frac{R-FP}{Q}, I = \frac{S-HQ}{R}$$

etc., then the series is equal to

$$\cfrac{B}{P + \cfrac{DP}{\cfrac{Q-D}{P} + \cfrac{FPQ}{\cfrac{R-FP}{Q} + \cfrac{H}{\cfrac{S-HQ}{R} + \cfrac{K}{\text{etc.}}}}}}$$

or

$$\cfrac{B}{P + \cfrac{DP}{Q - D + \cfrac{FPQ}{R - FP + \cfrac{HQR}{S - HQ + \cfrac{KRS}{\text{etc.}}}}}}.$$

If we were given the series

$$\frac{a}{p} - \frac{b}{q} + \frac{c}{r} - \frac{d}{s} + \frac{e}{t} - \text{etc.}$$

because

$$B = a, D = b : a, F = c : b, H = d : c, K = e : d\text{etc.}$$

and

$$P = p, Q = q : p, R = pr : q, S = qs : pr, T = prt : qs\text{etc.},$$

the sum of the series would be equal to

$$\cfrac{a}{p + \cfrac{b : a}{\cfrac{aq - bp}{app} + \cfrac{c : b}{\cfrac{p^2(br - cq)}{bqq} + \cfrac{d : c}{\cfrac{q^2(cs - dr)}{cp^2r^2} + \cfrac{e : d}{\cfrac{p^2r^2(dt - es)}{dq^2s^2} + \text{etc.}}}}}}$$

$$= \cfrac{a}{p + \cfrac{bp^2}{aq - bp + \cfrac{acqq}{br - cq} + \cfrac{bdrr}{cs - dr + \cfrac{cess}{dt - es + \text{etc.}}}}}$$

As an example, let's look at the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \text{etc.},$$

whose sum is $\log 2$ or $\int \frac{dx}{1+x}$, if after the integration we set $x = 1$. Therefore, it will be

$$a = b = c = d = \text{etc.} = 1, p = 1, q = 2, r = 3, s = 4 \quad \text{etc.}$$

and

$$p = 1, aq - bp = 1, br - cq = 1, cs - dr = 1 \quad \text{etc.}$$

Hence it will be

$$\int \frac{dx}{1+x} = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{4}{1 + \cfrac{9}{1 + \cfrac{16}{1 + \text{etc.}}}}}}$$

which, again, the value of this continued fraction is $\log 2$.

Many other similar series can be found after putting $x = 1$ after the integration:

$$\int \frac{dx}{1+x^3} = \cfrac{1}{1 + \cfrac{1^2}{3 + \cfrac{4^2}{3 + \cfrac{7^2}{3 + \cfrac{10^2}{3 + \text{etc.}}}}}} \qquad \int \frac{dx}{1+x^4} = \cfrac{1}{1 + \cfrac{1^2}{4 + \cfrac{5^2}{4 + \cfrac{9^2}{4 + \cfrac{13^2}{4 + \text{etc.}}}}}}$$

$$\int \frac{dx}{1+x^5} = \cfrac{1}{1 + \cfrac{1^2}{5 + \cfrac{6^2}{5 + \cfrac{11^2}{5 + \cfrac{16^2}{5 + \text{etc.}}}}}} \qquad \int \frac{dx}{1+x^6} = \cfrac{1}{1 + \cfrac{1^2}{6 + \cfrac{7^2}{6 + \cfrac{13^2}{6 + \cfrac{19^2}{6 + \text{etc.}}}}}}$$

So if we put $x = 1$ after the integration, then in general,

$$\int \frac{dx}{1+x^m} = \cfrac{1}{1+\cfrac{1^2}{m+\cfrac{(m+1)^2}{m+\cfrac{(2m+1)^2}{m+\cfrac{(3m+1)^2}{m+\text{etc.}}}}}}$$

And if m was fractional, then we would have

$$\int \frac{dx}{1+x^m} = \cfrac{1}{1+\cfrac{n}{m+\cfrac{(m+n)^2}{m+\cfrac{(2m+n)^2}{m+\cfrac{(3m+n)^2}{m+\text{etc.}}}}}}$$

Now let's consider $\int \frac{x^{n-1}dx}{1+x^m}$, which if we put $x = 1$ afterwards gives the series

$$\frac{1}{n} - \frac{1}{m+n} + \frac{1}{2m+n} - \frac{1}{3m+n} + \text{etc.}$$

Hence it will be

$$a = b = c = d = \text{etc.} = 1 \quad \text{and} \quad p = n, q = m+n, r = 2m+n, s = 3m+n\text{etc.}$$

Plugging these in, we get

$$\int \frac{x^{n-1}dx}{1+x^m} = \cfrac{1}{n+\cfrac{n^2}{m+\cfrac{(m+n)^2}{m+\cfrac{(2m+n)^2}{m+\text{etc.}}}}}$$

which can be seen coincides with the last one found.

Now we consider $\int \frac{x^{n-1}dx}{(1+x^m)^{\frac{p}{q}}}$, where after integrating and putting $x = 1$ gives the series

$$\frac{1}{n} - \frac{\mu}{v(m+n0} + \frac{\mu(\mu+v)}{1 \cdot 2v^2(2m+n)} - \frac{\mu(\mu+v)(\mu+2v)}{1 \cdot 2 \cdot 3v^3(3m+n)} + \text{etc.},$$

which compared to the general one yields

$$a = 1, b = \mu, c = \mu(\mu+v), d = \mu(\mu+v)(\mu+2v)\text{etc.},$$

$$p = n, q = v(m+n), R = 2v^2(2m+n), s = 6v^3(3m+n), t = 24v^4(4m+n)\text{etc.}$$

and

$$\begin{aligned}
aq - bq &= vm + (v-\mu)n, \\
br - cq &= \mu v(3v-\mu)m + \mu v(v-\mu)n, \\
cs - dr &= 2\mu v^3(\mu+v)(m(5v-2\mu) + n(v-\mu)), \\
dt - es &= 6\mu v^3(\mu+v)(\mu+2v)(m(7v-3\mu) + n(v-\mu)) \\
&\text{etc.}
\end{aligned}$$

After substituting these values and after some simplifications, and also setting $A = v - \mu$ (not related to variables previously mentioned and only for formatting), we get

$$\int \frac{x^{n-1}dx}{(1+x^m)^{\frac{u}{v}}} =$$

$$\cfrac{1}{n + \cfrac{\mu n^2}{vm + An + \cfrac{v(\mu+v)(m+n)^2}{(3v-\mu) + An + \cfrac{2v(\mu+2v)(2m+n)^2}{(5v-2\mu)m + An + \cfrac{3v(\mu+3v)(3m+n)^2}{(7v-3\mu)m + An + \cdots}}}}}$$

If we let $\mu = 1$ and $v = 2$, then it becomes

$$\int \frac{x^{n-1}dx}{\sqrt{1+x^m}} = \cfrac{1}{n + \cfrac{n^2}{2m + n + \cfrac{6(m+n)^2}{5m + n + \cfrac{20(2m+n)^2}{8m + n + \cfrac{42(3m+n)^2}{11m + n + \cfrac{72(4m+n)^2}{14m + n + \text{etc.}}}}}}}$$

Now if we let $v = 1$ and μ be an integer, it will result

$$\int \frac{x^{n-1}dx}{(1+x^m)^2} = \cfrac{1}{n + \cfrac{2n^2}{m - n + \cfrac{1\cdot 3(m+n)^2}{m - n + \cfrac{2\cdot 4(2m+n)^2}{m - n + \cfrac{3\cdot 5(3m+n)^2}{m - n + \cfrac{4\cdot 6(4m+n)^2}{m - n + \text{etc.}}}}}}}$$

$$\int \frac{x^{n-1}dx}{(1+x^m)^3} = \cfrac{1}{n + \cfrac{3n^3}{m - 2n + \cfrac{1 \cdot 4(m+n)^2}{-2n + \cfrac{2 \cdot 5(2m+n)^2}{-m - 2n + \cfrac{3 \cdot 6(3m+n)^2}{-2m - 2n + \cfrac{4 \cdot 7(4m+n)^2}{-3m - 2n + \text{etc.}}}}}}}$$

These do not converge, but diverge because of the negative quantities.

Continued fractions and so obviously Euler's work on continued fractions are important as continued fractions present a way to approximate numbers in a rational form, and many properties of such numbers could be found in continued fraction form.

References

[EA18] Leonhard Euler and Alexander Aycock. Observations on continued fractions, 2018.