Euler's Formula

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Euler's formula.

As is well known,

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

One proof of this is carried out with the use of the Taylor expansions of cosine and sine. The Taylor expansion of sine is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}.$$

Writing this out for later clarity,

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots .$$

Similarly, the Taylor expansion of cosine is given by

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

which is

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots .$$

Theorem 1.

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

Proof one. Consider the Taylor expansion of e^z which is

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$$= 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots .$$

Then, let $z = i\theta$. This series becomes

$$\sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}$$

$$= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \cdots$$

$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \cdots .$$

Looking briefly at the expansions of $\sin x$ and $\cos x$,

$$
\begin{aligned}
&= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \cdots \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots\right) + \left(i\theta - \frac{i\theta^3}{3!} + \frac{i\theta^5}{5!} \cdots\right) \\
&= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \\
&= \cos\theta + i\sin\theta.
\end{aligned}
$$

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A secondary proof relies on the following theorem.

Proof two. Consider the following

$$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n.$$

Much like before, let $x = i\theta$ so

$$e^{i\theta} = \lim_{n\to\infty}\left(1 + \frac{i\theta}{n}\right)^n.$$

Then, note that

$$1 + \frac{ix}{n} = \sqrt{1 + \frac{x^2}{n^2}}\left(\cos\left(\tan^{-1}\left(\frac{x}{n}\right)\right) + i\sin\left(\tan^{-1}\left(\frac{x}{n}\right)\right)\right).$$

From this,

$$(1 + \frac{ix}{n})^n = (\sqrt{1 + \frac{x^2}{n^2}})^n\left(\cos\left(n\tan^{-1}\left(\frac{x}{n}\right)\right) + i\sin\left(n\tan^{-1}\left(\frac{x}{n}\right)\right)\right).$$

Since

$$\lim_{n\to\infty}\left(1 + \frac{x^2}{n^2}\right)^{\frac{n}{2}} = 1$$

and

$$\lim_{n\to\infty} n\tan^{-1}\left(\frac{x}{n}\right) = x,$$

$$\lim_{n\to\infty}\left(1 + \frac{ix}{n}\right)^n = \cos x + i\sin x.$$

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Theorem 2.

$$e^{i\pi} + 1 = 0.$$

Proof. This is just plugging in π. It is

$$\begin{aligned} e^{i\pi} &= \cos\pi + i\sin\pi \\ &= -1 + 0. \end{aligned}$$

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Considering the formula, note that

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}.$$

This can be expanded to show it is true

$$\begin{aligned} \cos x &= \frac{e^{ix} + e^{-ix}}{2} \\ &= \frac{\cos x + i\sin x + \cos x - i\sin x}{2} \\ &= \frac{2\cos x}{2}. \end{aligned}$$

Similarly,

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}.$$

This can be shown to be true in the same way.

Angle sum and difference identities.

This can be used to show the angle sum and difference identities.

Theorem 3.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

Proof. Noting that $\cos x = \frac{e^{ix}+e^{-ix}}{2}$,

$$\begin{aligned} \cos(\alpha+\beta) &= \frac{e^{i(\alpha+\beta)} + e^{-i(\alpha+\beta)}}{2} \\ &= \frac{e^{i\alpha+i\beta} + e^{-i\alpha-i\beta}}{2} \\ &= \frac{e^{i\alpha}e^{i\beta} + e^{-i\alpha}e^{-i\beta}}{2} \\ &= \frac{(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) + (\cos\alpha - i\sin\alpha)(\cos\beta - i\sin\beta)}{2} \\ &= \frac{2(\cos\alpha\cos\beta - \sin\alpha\sin\beta)}{2} \\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta. \end{aligned}$$

Similarly for $\cos(\alpha - \beta)$. ∎

This can also be used to prove the $\sin x$ analogue.

Note also that

$$\cos x = \frac{e^i x + e^{-ix}}{2}$$

means

$$\begin{aligned} &\frac{d}{dx}\frac{1}{2}(e^i x + e^{-ix}) \\ &= \frac{ie^{ix} - ie^{-ix}}{2} \\ &= \frac{i\cos x + i^2 \sin x - i\cos x + i^2 \sin x}{2} \\ &= \frac{-2\sin x}{2} \\ &= -\sin x \end{aligned}$$

as expected.

Expansion of powers of cosine.

Let $u = \cos\theta + i\sin\theta$ and $v = \cos\theta - i\sin\theta$. Then

$$u^n = \cos n\theta + i\sin x \text{ and } v^n = \cos nx - i\sin nx.$$

Consider the following problem, proposed in [E246]. Firstly, $2^n \cos^n x = (u+v)^n$. This can be expanded

$$2^n \cos^n x = u + nu^{n-1}v + \frac{n(n-1)}{2}u^{n-2}v^2 + \cdots.$$

Similarly,

$$2^{n+1}\cos^n x = u^n + v^n + \binom{n}{n-1}(u^{n-2} + v^{n-2})uv + \binom{n}{n-2}(u^{n-4} + v^{n-4})u^{n-2}v^2 + \cdots.$$

Noting that $u^n + v^n = 2\cos nx$, this is the same as

$$2^n \cos^n x = \cos nx + n\cos(n-2)x + \frac{(n)(n-1)}{2!} + \cdots.$$

Considering

$$\sin^m\theta\cos^n\theta = \frac{(u-v)^m}{2^m i^m}\frac{(u_v)^n}{2^n}$$

since the power m is of the form $4a+1$,

$$i^m = i^{4a+1} = i.$$

Therefore,

$$\begin{aligned} 2^{m+n} i \sin^m x \cos^n x &= (u+v)^m (u+v)^n \\ &= -(v-u)^m (v+u)^n. \end{aligned}$$

Then,

$$2^{m+n}\sin^m x\cos^m x = -singx - A\sin(g-2)x - B\sin(g-4)x - \cdots$$

with

$$A = f$$
$$2B = fA - g$$
$$3C = fB - (g-1)A$$
$$4D = fC - (g-2)B$$

and so on. Consider $\frac{\sin x}{\cos x}$, which has $m = 1$ and $n = -1$ as well as $f = -2$ and $g = 0$. Then,

$$A = -2$$
$$2B = -2(-2) - 0 = 4$$
$$B = 2$$
$$3C = (-2)(2) - (-1)(-2) = -6$$
$$C = -2$$
$$4D = (-2)(-2) - (-2)(2)$$
$$D = 2$$

and so on. Since $m = 1$,

$$\frac{\sin x}{\cos x} = 2\sin 2x - 2\sin 4x + 2\sin 6x - 2\sin 8x + \cdots.$$