ON A NOTABLE ADVANCEMENT OF DIOPHANTINE ANALYSIS

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1. Abstract

This written paper is a synopsis and simplification of Leonhard Euler's work in his paper [Eul18a]. Building off of his previous paper [Eul18b], Euler uses the general fourth power formula and applies simplifications and substitutions to continue to reduce the formula. Using the quadratic formula, he then finds the roots that satisfy the formula, and then forms a series of the candidate solutions. Using the series, he goes back to the original formula and beautifully reduces the expression, and considers three separate examples with potential solutions from there.

2. The Problem of Fourth Degree

Prior to Euler's analysis of Diophantine Equations, there was no rigorous, straightforward way of finding multiple solutions to fourth-power formulas that must equal a square. There was too much work involved for humans and primitive machines to brute force solutions. Euler, however, observed that any fourth-power formula equivalent to a square must be in the form:

$$a^2x^4 + 2abx^3y + cx^2y^2 + 2bdxy^3 + d^2y^4$$

which can then be reduced to:

$$(ax^2 + bxy + dy^2)^2 + x^2y^2(c - b^2 - 2ad)$$

taking $c - b^2 - 2ad = mn$, a substitution is made so:

$$(ax^2 + bxy + dy^2)^2 + x^2y^2(mn)$$

is equivalent to a square, which will satisfy:

$$ax^2 + bxy + dy^2 = \lambda(mp^2 - nq^2)$$

with

$$xy = 2\lambda pq$$

Now, $(2\lambda(mp^2+nq^2))^2$ turns out to be a square, and there are many different representations of the expression, using the factors of mn.

Euler considered the numbers m, n, p, q as fractions, as to letting y equal to 1 and writing $x = 2\lambda pq$, to get the following equation through substitution to the previous equation:

$$4\lambda^2ap^2q^2 + 2\lambda bpq + d = \lambda mp^2 - \lambda nq^2$$

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Which is a quadratic equation, with respect to both p and q, and we get the following roots in these formulas;

$$p = \frac{-\lambda bq \pm \sqrt{\lambda md + \lambda^2 q^2(b^2 - 4ad + mn) - 4\lambda^3 anq^4}}{4\lambda^2 aq^2 - \lambda m}$$

and

$$q = \frac{-\lambda bp \pm \sqrt{-\lambda nd + \lambda^2 q^2(b^2 - 4ad + mb) + 4\lambda^3 amp^4}}{4\lambda^2 ap^2 - \lambda n}$$

3. A Series for the Solutions

Euler assigned that

$$p + p' = \frac{-2bq}{4\lambda aq^2 - m}$$

and

$$q + q' = \frac{-2bp}{4\lambda ap^2 + n}$$

with p and q being the already discovered values, and p' and q' the new values, with q' being the value given when p' is substituted into the formula in place for p. A series can be generated of the following: $q, p, q', p', q'', \ldots$, from the formulas above, with:

$$p' = \frac{-2bq}{4\lambda aq^2 - m} - p$$

$$p'' = \frac{-2bq'}{4\lambda aq'^2 - m} - p'$$

$$p''' = \frac{-2bq''}{4\lambda aq''^2 - m} - p''$$

and vice versa for q, with $q = p'$ and $-m = n$:

$$q' = \frac{-2bp'}{4\lambda ap'^2 + n} - q$$

$$q'' = \frac{-2bp''}{4\lambda ap''^2 + n} - q'$$

$$q''' = \frac{-2bp'''}{4\lambda ap'''^2 + n} - q''$$

With $y = 1$ in the original formula, the following expression is generated:

$$2\lambda pq, 2\lambda q'p, 2\lambda p'q', 2\lambda q''p', \ldots$$

and

$$2\lambda qp, 2\lambda p'q, 2\lambda q'p', 2\lambda p''q', \ldots$$

Looking back at the original formula:

$$a^2x^4 + 2abx^3y + cx^2y^2 + 2bdxy^3 + d^2y^4$$

and setting the root to $ax^2 - bx - d$, we get:

$$x = \frac{4bd}{b^2 - 2ad - c} = \frac{-4bd}{mn + 4ad}$$

with $c = mn + b^2 + 2ad$. Now that we have a suitable value of x with $y = 1$, the variables p and q can be found from:

$$ax^2 + bx + d = \lambda(mp^2 - nq^2)$$

which becomes:

$$\frac{ax^2 + bx + d}{x} = \frac{mp^2 - nq^2}{2pd}$$

with $x = 2\lambda pq$. Thus, by plugging in $A = \frac{ax^2+bx+d}{x}$, we get that

$$2Apq = mp^2 - nq^2$$

from which we get $\frac{p}{q} = \frac{A+\sqrt{A^2+mn}}{m}$, and we get $\frac{f}{g} = \frac{p}{q}$, where we can rightfully assume that $f = p$ and $g = q$, and the rest of the work will be carried out by examples. Substituting into the general formula $\alpha A^4 \pm \beta B^4$ equivalent to a square, we can take $\frac{A}{B} = C$, so the formula simplifies to $\alpha C^4 \pm \beta =$. Letting $C = \frac{1+x}{1-x}$, we get $\alpha + \beta = a^2$, giving us the following formula:

$$a^2 + 4(\alpha - \beta)x + 6a^2x^2 + 4(\alpha - \beta)x^3 + a^2x^4 =$$

4. Example 1

Euler first considered the formula $2A^4 + B^4$ equaling a square. This formula is the *same* as the one derived from have two numbers A and B with $A + B$ equaling a square and $A^2 + B^2$ a fourth power, as presented in [Eul18b]. Therefore, having set $\frac{A}{B} = C$, so that $2C^4 - 1$ is a square, then $\alpha = 2$ and $beta = -1$, from which $\alpha + \beta = 1 = a^2$, so $a = 1$, we can plug in $C = \frac{1+x}{1-x}$ back into the equation and get the following:

$$1 + 12x + 6x^2 + 12x^3 + x^4$$

or

$$(1 + 6x + x^2)^2 - 32x^2$$

is equal to a square. Then, substituting the values as follows

$$1 + 6x + x^2 = \lambda(p^2 + 8q^2)$$

and

$$x = \lambda pq$$

we get the equation

$$1 + 6\lambda pq + \lambda^2p^2q^2 = \lambda p^2 + 8\lambda q^2$$

and then, using the formula for roots for p and q, we get the following:

$$p = \frac{-3\lambda q \pm \sqrt{8\lambda^3q^4 + \lambda}}{\lambda^2q^2 - \lambda}$$

and

$$q = \frac{-3\lambda p \pm \sqrt{\lambda^3p^4 + 8\lambda}}{\lambda^2p^2 - 8\lambda}$$

then, *regardless* of how the $\pm$ is assigned, we get

$$p + p' = \frac{-6q}{\lambda q^2 - 1}$$

$$q + q' = \frac{-6p}{\lambda p^2 - 8}$$

Now Euler looked at the values of p and q that come forth from these functions. Taking $\lambda = 0$, we quickly see $q = 0, p = 1$ fulfills the requirements. If we take $q = 1$, we get $p = \frac{-3\pm3}{1-1}$, but in the original case $q = 1$ gives $p = \frac{7}{6}$. The relationship of the derived values can be written as

$$p + p' = \frac{-6q}{q^2 - 1}$$

$$q + q' = \frac{-6p}{p^2 - 8}$$

And we can set up a series for the numbers $q, p, q', p', q'', etc.$

$$q' = \frac{-6p}{\lambda p^2 - 8} - q$$

$$q'' = \frac{-6p'}{\lambda p'^2 - 8} - q'$$

$$\cdots$$

and vice versa for p:

$$p' = \frac{-6q}{q^2 - 1} - p$$

$$p'' = \frac{-6q'}{q'^2 - 1} - p'$$

$$\cdots$$

Therefore, from the series where $q = 0$ and $p = 1$, we get the following series

$$0, 1, \frac{6}{7}, \frac{239}{13}, \ldots$$

From this series, we can take values two at a time to plug into x, and into $C = \frac{1+x}{1-x}$, we get this series for x: $0, \frac{6}{7}, \frac{1434}{91}, \ldots$, and the series for C: $1, 13, -\frac{1525}{1343}, \ldots$. Then, looking at the other of $q = 1$ and $p = \frac{7}{6}$, we find the series of $p, q, p', q', \ldots$ to be: $1, \frac{7}{6}, \frac{13}{239}, \ldots$ from which we can see that the previous series have already exhausted *all* solutions, so it is not necessary to solve for the latter case.

5. Example 2

Then, Euler considered the formula $3A^4 + B^4$ equals a square. This formula can be fulfilled with $3C^4 + 1$ equaling a square, and it is not hard to observe that $C = 0, C = 1, C = 2$ fulfills this equation. Since $\alpha = 3$ and $\beta = 1$ here, after setting $C = \frac{1+x}{1-x}$, we get the formula $4 + 8x + 24x^2 + 8x^3 + 4x^4$, which, when divided by 4, yields:

$$1 + 2x + 6x^2 + 2x^3 + x^4$$

The formula can be represented as $(1 + x + x^2)^2 + 3x^2$, into which, with $x = 2\lambda pq$, we can substitute and get:

$$1 + x + x^2 = \lambda(p^2 - 3q^2)$$

From which the following is derived:

$$1 + 2\lambda pq + 4\lambda^2 p^2 q^2 = \lambda p^2 - 3\lambda q^2$$

Taking $\lambda = 1$ and $q = \frac{1}{2}$, the equation gives $p = -\frac{7}{4}$, and now we can evaluate the roots of the quadratic to be:

$$p = \frac{-\lambda q \pm \sqrt{\lambda - 12\lambda^3 q^4}}{4\lambda^2 q^2 - \lambda}$$

and

$$q = \frac{-\lambda p \pm \sqrt{4\lambda^3 p^4 - 3\lambda}}{4\lambda^2 p^2 + 3\lambda}$$

Thus, the solutions give

$$p + p' = \frac{-2\lambda pq}{4\lambda^2 q^2 - \lambda}$$

and

$$q + q' = \frac{-2\lambda p}{4\lambda^2 q^2 + 3\lambda}$$

Since we already have the $\lambda = 1, q = \frac{1}{2}, p = -\frac{7}{4}$ case, our series of $q, p, q', p', q'', etc.$ are formed from $p + p' = \frac{-2pq}{4q^2-1}$, and $q + q' = \frac{-2p}{4q^2+3}$, which generate the series $\frac{1}{2}, -\frac{7}{4}, -\frac{33}{122}, \ldots$. Then, when $x = 2pq$, we get $x = -\frac{7}{4}$ and $x = \frac{231}{448}$, giving $C = -\frac{3}{11}$. For then, the answer is

$$\sqrt{3C^4 + 1} = \frac{122}{121}$$

6. Example 3

Consider $\frac{3A^4 - B^4}{2}$: since the square now has to be $\frac{3}{2}C^4 - \frac{1}{2}$, we get $\alpha = \frac{3}{2}, \beta = -\frac{1}{2}$, and therefore $a^2 = \alpha + \beta, a = 1$, and $\alpha - \beta = 2$. The formula of x becomes:

$$1 + 8x + 6x^2 + 8x^3 + x^4$$

or

$$(1 + 4x + x^2)^2 - 3(2x)^2$$

Here, Euler set $x = \lambda pq$, so

$$1 + 4x + x^2 = \lambda(p^2 + 3q^2)$$

which produces the following equation in p and q

$$1 + 4\lambda pq + \lambda^2 p^2 q^2 = \lambda p^2 + 3\lambda q^2$$

First assuming $\lambda = 1$ and $q = 1$, the equation gives $p = \frac{1}{2}$. Then, taking $\lambda = 3$ and $p = 1$, it gives $q = \frac{1}{6}$, and we can split the rest of the evaluation into two cases In the first case with $\lambda = 1$, let $x = pq$, and the equation is now:

$$p^2(q^2 - 1) + 4pq + 1 = 3q^2$$

The sum of the roots is $p + p' = \frac{-4q}{q^2-1}$. Similarly, $q + q' = \frac{-4p}{p^2-3}$. From here, the series $p, q, p', q', \ldots$ will evaluate to be $1, \frac{1}{6}, -\frac{3}{11}, -\frac{47}{84}, \ldots$. Since $x = \lambda pq = 3pq$, we get $x = \frac{1}{2}, -\frac{-3}{22}, -\frac{141}{308}$.

7. Conclusion

With this, Euler concludes that he has "more than abundantly indicated the widest use of this method", and the proof comes to an end.

References

[Eul18a] Leonhard Euler. *De insigni promotione Analysis Diophantaeae*. 2018.

[Eul18b] Leonhard Euler. *Solutio problematis Fermatiani de duobus numeris, quorum summa sit quadratum, quadratorum vero summa biquadratum, ad mentem illustris La Grange adornata*. 2018.