EULER'S PERFECT SQUARES

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1. The Officer Problem

Some say it was Catherine the Great that first posed the question to Euler while he was visiting the St. Petersburg Academy late in his life: *Can one arrange a group of thirty-six officers of six different ranks, taken from six different regiments, and arranged in a square in a way such that in each row and column there are six officers, each of a different rank and regiment?*. Even though Euler was never able to rigorously prove that the answer is in fact no, there is no way to arrange the group of 36, his paper [Eul82] went a ways to prove its falsehood. In fact, it was not until 1901 that a proof for the afore mentioned problem was published, and even it was only proven by taking every single possible combination, of which there are 9408. In this paper, I will discuss the method Euler used to prove a square with side length 6 will never yield a perfect Graeco-Latin square. Euler proved that squares of single step construction can not be made when the side length is an even number, and that double step squares can not be constructed when the side length is in the form $4n+2$, along with providing interesting logical insights into why he believed that side length $n = 6$ is impossible in general. At the end of his paper, Euler conjectured that it is impossible to build a Latin square for any value in the form $4n+2$ based on his observations of six-sided and two-sided Graeco-Latin squares. Although he was ultimately unable to rigorously to prove this statement in the case of the six-sided square in the officer problem, his logical process and mathematical persistence are on full display in this paper.

2. Graeco-Latin Squares

Definition 2.1. A Latin square is a square with side length n, such that each row and column contains the numbers $1, 2, 3 \cdots, (n-1), n$ exactly once.

There are several different ways to construct Latin squares, the easiest being by the step method, in which each row is slightly altered based on the row before it. The most simple of any Latin squares is the single step Latin square, which is constructed by starting with the numbers 1 through n in ascending order in the first row, and creating each successive row by moving the number in the first column to the last column, and shifting all other terms to the left one row. The single step Latin square for a side length $n = 6$ is

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 1 & 2 & 3 & 4 & 5. \end{array}$$

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Latin squares of step length 2 and greater can also be created by combining numbers into sets of the same size and changing the order of the numbers from line to line appropriately. The double step and triple step Latin squares for $n = 6$ are

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 3 & 6 & 5 & 2 & 1 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 5 & 2 & 1 & 4 & 3 \end{array} \qquad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 1 & 5 & 6 & 4 \\ 3 & 1 & 2 & 6 & 4 & 5 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & 6 & 4 & 2 & 3 & 1 \\ 6 & 4 & 5 & 3 & 1 & 2. \end{array}$$

It is important to note that multi step Latin squares can only be formed correctly when the step size of the square is a divisor of the total side length, n. In other words, double step Latin squares can only be used to construct squares where n is an even number, triple step squares only work when n is a multiple of three, and so on.

Definition 2.2. A Graeco-Latin square is a square with side length n such that each term in the square is represented as p^q where $p, q \in (1, n)$, each row and column contains all numbers from 1 to n as both a base and exponent, and each variation of paired base and exponent appears exactly once in the square.

Euler decided to approach the building of a Graeco-Latin in a different way. He noticed that every exponent in the square was joined to each base once, and only appeared once in each row and column. Therefore, he introduced the principal of a guiding formula for an exponent, which he used extensively to create Graeco-Latin squares.

Definition 2.3. The guiding formula for any exponent, n, is the route that n takes through the rows of the square with respect to the base of the exponential.

For instance, take the Graeco-Latin square with side length 3

$$\begin{array}{ccc} 1^1 & 2^3 & 3^2 \\ 2^2 & 3^1 & 1^3 \\ 3^3 & 1^2 & 2^1. \end{array}$$

The guiding formula for the exponent 1 in this square would be the sequence 1, 3, 2, where 1 appears as the exponent over base 1 in row 1, over base 3 in row 2, and over base 2 in row 3. Each exponent in Euler's square had to have its own guiding formula, and the guiding formulas had to agree with each other in such a way that every exponent is represented once per base, once per column, and once per row.

3. Single-Step Squares

Theorem 3.1. *There is no possible single step Latin square construction for any side length n, such that n is even.*

Proof. Let

$$1 \quad a \quad b \quad c \quad d \quad e \quad etc$$

be a guiding formula for the exponent 1, where the letters $a, b, c, d, e, etc.$ represent the numbers from 2 to n that make up the bases of the exponent 1 in any order. Let $\alpha, \beta, \gamma, \delta, \epsilon, etc$ denote the column in which the bases are, such that they to take all the values from 2 to n. The base of 1 in row 2 is denoted by a, and is in column α. Since this is a single step Latin square, each row has been shifted to the left compared to the previous row. Therefore

$$\alpha \equiv a + 1 \pmod{n}$$

because in the second row, the value of the base will be one less than the index of the column. Similarly,

$$\beta \equiv b + 2 \pmod{n}$$

since in the third row the bases are shifted 2 places over relative to the index of the column. More generally, any Greek number p, denoting the column of index p, will be congruent to its Latin number r plus the value $(m - 1)$, where m is the row that r is in. Since we know that

$$\alpha + \beta + \gamma + \delta + \epsilon + \cdots = a + b + c + d + e + \cdots$$

and

$$\alpha + \beta + \gamma + \delta + \epsilon + \cdots = ((a + b + c + d + e + \cdots) + (1 + 2 + 3 + \cdots + (n - 1))) \pmod{n}$$

we can set

$$(1 + 2 + 3 + \cdots + (n - 1)) = \frac{1}{2}n(n + 1) = 0 \pmod{n}.$$

For $\frac{1}{2}n(n + 1) = 0 \pmod{n}$ to be true, $\frac{1}{2}n(n + 1)$ must be a multiple of n, which means we can change the equation to

$$qn = \frac{1}{2}n(n + 1)$$

where q is a whole number. Simplifying, we get $q = \frac{1}{2}(n - 1)$ which means that $\frac{1}{2}(n - 1)$ must be a whole number. Since $\frac{1}{2}(n - 1)$ will only be a whole number if $(n - 1)$ is even, or n is odd, n cannot be an even number. ∎

Euler wished to develop a method which he could use to construct several squares for the same side length without having to verify every single character in the square by hand. To do this, Euler let

$$1 \quad a \quad b \quad c \quad d \quad e \quad etc$$

be a guiding formula corresponding to the exponent 1, where the index t is defined as the row number that the base x exists in. For example, when $t = 1$, $x = 1$ where t refers to the first row and x refers to the base number in the guiding formula for the exponent 1. Similarly, when $t = 3$, $x = b$ where b is the base that is raised to the power of 1 in row 3. When we give t all the values from 1 to n, x also receives all values from 1 to n, as is the definition of a Graeco-Latin square. Euler then let

$$1 \quad A \quad B \quad C \quad D \quad E \quad Etc.$$

be a different guiding formula for a square with same side length n as the previous guiding formula. Euler defined X and T in the same fashion, where X is the value of the base in row T. Euler wished for a rule to derive one guiding formula for the other, and so he naturally set $T = x$ and $X = t$ which creates new values for the guiding formula and continues to satisfy the requirements of exponential representation for Graeco-Latin squares.

Example 3.1. Let us take the single step Latin square with side length $n = 7$,

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 3 & 4 & 5 & 6 & 7 & 1 \\ 3 & 4 & 5 & 6 & 7 & 1 & 2 \\ 4 & 5 & 6 & 7 & 1 & 2 & 3 \\ 5 & 6 & 7 & 1 & 2 & 3 & 4 \\ 6 & 7 & 1 & 2 & 3 & 4 & 5 \\ 7 & 1 & 2 & 3 & 4 & 5 & 6 \end{array}$$

and take a random guiding formula for the base 1, which will be our x

$$x = 1 \quad 4 \quad 2 \quad 7 \quad 6 \quad 3 \quad 5$$

$$t = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7.$$

We can now switch the values of the table to get a new X and T

$$X = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$$

$$T = 1 \quad 4 \quad 2 \quad 7 \quad 6 \quad 3 \quad 5.$$

Now, in order to get our new guiding formula in the correct order of terms, we can rearrange our (X, T) pairs in order to get the T values to take the form of the natural numbers. By rearranging the terms, we get

$$X = 1 \quad 3 \quad 6 \quad 2 \quad 7 \quad 5 \quad 4$$

$$T = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$$

our second guiding formula. However, this was not the only formula for x and t Euler discovered that allowed him to build new guiding formulas, in fact he found 10 more, the second of which is given by

$$T = t \qquad X = 1 + t - x.$$

Consider the same Latin square and guiding formula from the previous example:

$$x = 1 \quad 4 \quad 2 \quad 7 \quad 6 \quad 3 \quad 5$$

$$t = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7.$$

Finding the sequence for T is easy, as it is simply $1, 2, 3, 4, 5, 6, 7$, but finding values for X is harder. Whenever $x \geq t + 1$, we get a non-positive value for X, which occurs when $x = 4$, $x = 7$, and $x = 6$. To assign values to X at the values of X not in the set $(1, n)$, when $x = 4, 7, 6$, simply reduce the value in $\pmod{n}$ until it is between 1 and n, so $7 \equiv 0 \pmod{n}$, $-1 \equiv 6 \pmod{n}$, and so forth. Thus, our sequences for X and T are:

$$X = 1 \quad 6 \quad 2 \quad 5 \quad 7 \quad 4 \quad 3$$

$$T = 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7$$

and since our T values are already in the correct order, our third guiding formula is

$$1 \quad 6 \quad 2 \quad 5 \quad 7 \quad 4 \quad 3.$$

Note: Interestingly enough, using either formula on the same sequence, and then the resulting sequence will give you the sequence you started with. Euler instead noticed that these two rules for constructing new guiding formulas work well together, and used them alternately to construct lists of guiding formulas, all derived from a single guide. In this case, alternating formulas on each derived series results in the discovery of 11 new series before you get your starting formula again.

4. Multi-step squares

We have already seen Euler prove that single step Latin squares are impossible to build for side length 6, as no square with an even side length can be constructed for single step squares. In order to prove no Graeco-Latin square is possible for $n = 6$, Euler next turned his attention to double step Latin squares.

Theorem 4.1. *A double step Latin square can only be built if* n *is a multiple of 4.*

Proof. Take a double step Latin square with side length n. Let us define t as the vertical index of x (the row that x is in), and u as the horizontal index of x (the column that x is in). By examining the square we can come to the following conclusions about the relationship between x, u, and t. When either u or t is odd, the formula for x is given by

$$x = t + u - 1,$$

and when both u and t are even, we have

$$x = t + u - 3.$$

Now let the series $(a, b, c, d, e, etc.)$ be the guiding formula for any exponent, r, in our double step square, and let $(\alpha, \beta, \gamma, \delta, \epsilon, etc.)$ be the corresponding sequence of values of u, or column indices, and let t, the rows, be represented by $(1, 2, 3, 4, 5, etc.)$. The first base term for r will always be $a = \alpha$, as the first row in our square simply counts up from 1 to n. Our second term, b, will have two equations to represent it, one for the case in which β is even, which would be the equation $b = \beta - 1$, and one for the odd value of β, $b = \beta + 1$. For term c, we know that $t = 3$ which satisfies our condition for formula one, giving us $c = \gamma + 2$ for c. d again gives us two possible equations, $d = \delta + 3$ and $d = \delta + 1$. For term e, as is the case for every other guiding term, t is odd, so $e = \epsilon + 4$. This pattern will continue forever, with every other term from the guiding formula having two case formulas for its value. We can observe that if the alternating values are all even, our formulas for each term are regular:

$$a = \alpha, \quad b = \beta + 1, \quad c = \gamma + 2, \quad d = \delta + 3, \quad etc..$$

Given the fact that $a + b + c + d + \cdots = \alpha + \beta + \gamma + \delta + \cdots \pmod{n}$, the formula for the difference of the two sequences must be some multiple of n, similar to our previous proof. Since we are dealing with a double step Latin squares however, we can set $n = 2m$, as n must be an even number, and therefore divisible by 2:

$$\frac{1}{2}n(n-1) = qn, \qquad m(2m-1) = 2qm.$$

We can then simplify by dividing by m and solve for x

$$2m - 1 = 2q, \qquad q = m - \frac{1}{2},$$

but since both m and q are defined as whole numbers, this equation cannot be true, which means that it is impossible to find a guide for the square if all the alternating letters are odd. Now let us assume that out of the alternating numbers, y of them have odd values of u. Since the difference between an even value of u and an odd value of u is 2, our new equation can be found by subtracting $2y$ from our equation where all alternating u values are odd, which gives us:

$$\frac{1}{2}n(n-1) - 2y = qn.$$

By letting $n = 2m$ as we did in the earlier equation we get

$$m(2m-1) - 2y = 2qm,$$

which we can see will only be true if m is even, as $2y$ and $2qm$ will always be even for any whole value of y, q, m, which means that $m(2m-1)$ must also be even. Therefore we can set $m = 2k$ and $n = 4k$ and simplify our equation

$$2k(4k-1) - 2y = 4qk, \qquad k(4k-1) - y = 2qk, \qquad y = k(4k-2q-1).$$

There are n different values of u, and y only counts some fraction of every other u value, so $y \leq \frac{1}{2}n$ or $y \leq 2k$ must be the case. Since our equation sets y as equal to $(4k-2q-1)$ times m, and $(4k-2q-1)$ has to be an odd number because all the coefficients are even numbers,

$$(4k-2q-1) = 1,$$

which means that we can solve for other variables, and get

$$y = m, \qquad q = 2m - 1.$$

This means that for there to exist a guiding formula for our double step square, it must have the properties that half of the alternating letters $(\beta, \gamma, \epsilon, etc.)$, or $\frac{1}{4}n = m$ of the total letters must be even, and n must be divisible by four, which proves that any whole number value of n such that $n = 4m + 2$ $(2, 6, 10, 14 \cdots)$, will never yield a guide for a double step square. ■

5. Euler's Conjecture

Euler knew that there were many Latin square possibilities that did not take the form of a n-step square. For instance, one can make a simple change to any square, formed using an n-step method, by finding a rectangle inside the square where opposite corners are the same numbers. To illustrate this, consider the single-step Latin square with $n = 6$ and it's rectangular transformation:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & \underline{3} & 4 & 5 & \underline{6} & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & \underline{6} & 1 & 2 & \underline{3} & 4 \\ 6 & 1 & 2 & 3 & 4 & 5 \end{array} \qquad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & \underline{6} & 4 & 5 & \underline{3} & 1 \\ 3 & 4 & 5 & 6 & 1 & 2 \\ 4 & 5 & 6 & 1 & 2 & 3 \\ 5 & \underline{3} & 1 & 2 & \underline{6} & 4 \\ 6 & 1 & 2 & 3 & 4 & 5. \end{array}$$

As you can see, by switching the 3 and 6 in rows 2 and 5, Euler was able to create a completely new Latin square which actually allows several guiding formulas for all exponents, the list for the base 1 is

$$1 \quad 6 \quad 5 \quad 2 \quad 4 \quad 3, \qquad 1 \quad 6 \quad 5 \quad 3 \quad 2 \quad 4,$$

$$\begin{array}{cccccc} 1 & 4 & 6 & 2 & 3 & 5, \\ 1 & 5 & 4 & 3 & 6 & 2, \\ 1 & 3 & 4 & 6 & 2 & 5, \end{array} \qquad \begin{array}{cccccc} 1 & 4 & 2 & 5 & 6 & 3, \\ 1 & 5 & 2 & 3 & 6 & 4, \\ 1 & 3 & 6 & 5 & 4 & 2. \end{array}$$

Unfortunately for Euler, although there are plenty of guiding formulas for each base for him to chose from, there is no way to take one for each base in a way that they all agree with each other. Euler used this and other methods to create families of squares where determining the ability of one to create a perfect Graeco-Latin square would tell you whether all the other squares in the family could yield squares as well. After considering dozens of different squares constructed methodically and randomly, and examining their families of squares, Euler concluded that $n = 6$ could not yield a full Graeco-Latin square, but admitted that he did not know for sure. He stated that if one did exist, its form would be very irregular, and that that one would have no easy way of knowing without examining every possible square of $n = 6$ side length. Combined with the knowledge that a two by two square can obviously never create a complete square, Euler conjectured:

Conjecture 5.1. *A square with side length n in the form $n = 4m + 2$ will never yield a perfect Graeco-Latin square.*

Although he was correct in the cases of $m = 1, n = 2$ and $m = 2, n = 6$, he was incorrect in his claim about all larger numbers $(10, 14, 18, \cdots)$. In fact it took over 150 years to disprove his conjecture with the help of group theory and computers, better specifics of which can be found in [KS06], and although he was wrong in the end, his investigations were a step forward for the research into the power of Graeco-Latin squares.

References

[Eul82] Leonhard Euler. Recherches sur un nouvelle espéce de quarrés magiques. *Verhandelingen uitgegeven door het zeeuwsch Genootschap der Wetenschappen te Vlissingen*, pages 85–239, 1782.

[KS06] Dominic Klyve and Lee Stemkoski. Graeco-latin squares and a mistaken conjecture of euler. *The College Mathematics Journal*, 37(1):2–15, 2006.