INVESTIGATION ON SINGLE-STEP LATIN SQUARES

DIETER YANG

1. Introduction

We begin with a very old researched question consisting of a group of thirty-six officers of six different ranks, taken from six different regiments, and arranged in a square in a way such that in each row and column there are six officers, each of a different rank and regiment. Now is this actually possible? Such an arrangement is called a **latin square**. The Korean mathematician Choi Seok-jeong was the first to create such a square to create a magic square of order 9 in the 1700's. Prior to 1748, such an arrangement was known to be impossible, but no one has rigorously showed this fact. Then, Euler was able to show that for any even equal number of ranks and regiments, such a configuration is not possible for single step latin squares (when the rows get shifted over by 1 each time). But then what happens if there is an odd number of regiments and ranks? We are able to find various methods to generate more arrangements given an existing arrangement. Then with this, we are able to establish a bijection between these **latin squares** and magic squares. This helps us create some method to generate magic squares without having to trial and error multiple magic squares, especially when the numbers get very large.

2. What is a Magic Square

Definition 1. *(Magic Square) A magic square is when the sum of all the numbers in each row, column and both diagonals of some $n \times n$ square are equal.*

An important thing to note is that we will only be focusing on natural numbers inside of our magic squares.
If we take some n by n square, and we used the natural numbers from 1 to n^2, then we know that the sum of all the squares is simply $\frac{n^2(1+n^2)}{2}$, and as a result the sum of each row, column and diagonal is $\frac{n(1+n^2)}{2}$. With this, we can make the following chart that depicts the sum of each row and column in some $n \times n$ square filled with the first n^2 natural numbers.

Date: August 20, 2023.

Magic Square		
n	n^2	Sum of each row/column $\left(\frac{n(1+n^2)}{2}\right)$
1	1	1
2	4	5
3	9	15
4	16	34
5	25	65
6	36	111
7	49	175
8	64	260
9	81	369

Here, we will provide an example of a $n = 5$ magic square. (See Table 1)

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

We can see that each row, column and both diagonals add up to 65.

3. Euler's Latin Square

We begin by labeling each cell of the magic square as shown below

$$\begin{matrix} 1^1 & 1^2 & 1^3 & 1^4 & 1^5 \\ 2^1 & 2^2 & 2^3 & 2^4 & 2^5 \\ 3^1 & 3^2 & 3^3 & 3^4 & 3^5 \\ 4^1 & 4^2 & 4^3 & 4^4 & 4^5 \\ 5^1 & 5^2 & 5^3 & 5^4 & 5^5 \end{matrix}$$

Now, what Euler wanted to investigate was to determine some way to generate arrangements of these numbers such that each row and each column had one of each number in the base and in the exponent.
There are many different types of arrangements. The first type of arrangement we will look at are **Single Step Latin Squares**.

Definition 2. *(Single Step Latin Squares) We define this in a $n \times n$ magic square as*

$$\begin{matrix} 1 & 2 & \dots & n \\ 2 & 3 & \dots & 1 \\ 3 & 4 & \dots & 2 \\ \vdots & \vdots & \ddots & \vdots \\ n & 1 & \dots & n-1 \end{matrix}$$

Now, we can apply a similar concept for Double Step Latin Squares, Triple Step Latin Squares, ect.
For example, a Double Step Latin Square would look something like this:

1	2	3	4	...	n
2	1	3	4	...	1
3	4	5	6	...	2
4	3	6	5	...	1

Now, it is important to note that when we are doing these n-step squares, we need our magic square to have dimensions divisible by n.
If we look at an example of a quadruple step latin square, we would have the following cases.

I				II				III				IV			
1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
2	1	4	3	2	1	4	3	2	3	4	1	2	4	1	3
3	4	1	2	3	4	2	1	3	4	1	2	3	1	4	2
4	3	2	1	4	3	1	2	4	1	2	3	4	3	2	1

4. When will these squares exist?

Euler began looking at when $n = 2$. Then we get the following Latin Square

1	2
2	1

Then, if we try and place exponents, we see it is not possible. Therefore we know that for $n = 2$, this fails.

$$\begin{matrix} 1^2 & 2^1 \\ 2^2 & 1^1 \end{matrix} \qquad\qquad \begin{matrix} 1^1 & 2^1 \\ 2^1 & 1^2 \end{matrix}$$

What if $n = 3$? Then we see only has one instance working in the case of Single Step Latin Squares.

$$\begin{matrix} 1^1 & 2^3 & 3^2 \\ 2^2 & 3^1 & 1^3 \\ 3^3 & 1^2 & 2^1 \end{matrix}$$

When we investigate $n = 4$, we realize that again, using single step squares, this is impossible. This brings us to Euler's first proposed theorem of

Theorem 4.1. *For all cases where the number n is even, the Single Step Latin square will never have some possible configuration.*

But how would you approach a proof? The way Euler approached this was by looking at the bases that were raised to the first power in order to find some contradiction.
First, we will define some notation.

Definition 3. *We define the guiding formula for some number k is the sequence of numbers that are raised to the k-th power starting from the first column.*

Proof. WLOG, let the guiding formula for 1 be $1, a, b, c, d, e, \ldots$, where the letters denote the numbers between 2 and n. We now label the rows using greek letters $\alpha, \beta, \ldots$. Now, we can also establish a bijection between the greek letters and our guiding formula. We also know that the sum of the greek letters must be equal to the sum of letters in our guiding formula. As this is a single step latin square, we see that the numbers of the rows increase by 1 arithmetically. Therefore we have that

$$a = \alpha + 1, b = \beta + 2, c = \gamma + 1, \ldots$$

When we get a number above n, we will simply write down the result modulo n. Now if we take the sum of $\alpha + \beta + \gamma + \cdots = S$, then

$$a + b + c + d + \cdots = S + 1 + 2 + 3 + \cdots + n = S + \frac{n(n-1)}{2}$$

Since the difference between the greek letters and the latin letters is some multiple of n, we see

$$\lambda \times n = \frac{n(n-1)}{2}$$

or that

$$\lambda = \frac{n-1}{2}$$

And we know that λ is an integer so therefore n must be odd. ■

In the case of $n = 5$, we take our one step latin square

1	2	3	4	5
2	3	4	5	1
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

Now, if we find our guiding formula for the exponent 1, we see there are three possibilities.

(1) $1, 3, 5, 2, 4$
(2) $1, 4, 2, 5, 3$
(3) $1, 5, 4, 3, 2$

Now, notice that we can add 1 to all of these numbers to create the guiding formula for the exponent 2, and repeat this process again for 3, 4, and 5 by adding 1 again. This will give us the following three arrangements that satisfy our conditions.

I					II					III				
1^1	2^5	3^4	4^3	5^2	1^1	2^3	3^5	4^2	5^4	1^1	2^4	3^2	4^5	5^3
2^2	3^1	4^5	5^4	1^3	2^2	3^4	4^1	5^3	1^5	2^2	3^5	4^3	5^1	1^4
3^3	4^2	5^1	1^5	2^4	3^3	4^5	5^2	1^4	2^1	3^3	4^1	5^4	1^2	2^5
4^4	5^3	1^2	2^1	3^5	4^4	5^1	1^3	2^5	3^2	4^4	5^2	1^5	2^3	3^1
5^5	1^4	2^3	3^2	4^1	5^5	1^2	2^4	3^1	4^3	5^5	1^3	2^1	3^4	4^2

Now we have some kind of method to create these Latin Squares for some odd n. Now, our objective is that given some guiding formula for any single step square. We want a reliable method in creating more guiding formulas and in return create more arrangements. WLOG, let

$$1, a, b, c, \dots$$

be our first guiding formula. Then, let our new guided formula that we want to find be $1, A, B, C, D, \dots$. One such way would be by transposing our square. Another way would be to take any term X with index T. Now we can create a new guide by taking

$$T = t \qquad X = \alpha + t - x$$

for some index x and the t that corresponds to it. We see that for each value t takes from 1 to n, then for all α, $\alpha + tx$ will take on different values. With this in mind, Euler was able to obtain the following methods to create more guiding formulas.

	I	II	III	IV	V	VI
$T = t$	t	x	$1+t-x$	x	$1+t-x$	$1+x-t$
$X = x$	$1+t-x$	t	t	$1+x-t$	$2-x$	x
	VII	VIII	IX	X	XI	XII
	$2-x$	$1+x-t$	$2-x$	$2-t$	$2-t$	$2-t$
	$1+t-x$	$2-t$	$2-t$	$1+x-t$	$2-x$	$2-x$

Then, taking our new guiding formulas for exponent 1, we can simply repeatedly add 1 as we did prior to create guiding formulas for our $n \times n$ square. Here is an example of what occurs for $n = 7$.

$$\begin{array}{ccccccc}
1^1 & 2^a & 3^b & 4^c & 5^d & 6^e & 7^f \\
2^2 & 3^{a+1} & 4^{b+1} & 5^{c+1} & 6^{d+1} & 7^{e+1} & 1^{f+1} \\
3^3 & 4^{a+2} & 5^{b+2} & 6^{c+2} & 7^{d+2} & 1^{e+2} & 2^{f+2} \\
4^4 & 5^{a+3} & 6^{b+3} & 7^{c+3} & 1^{d+3} & 2^{e+3} & 3^{f+3} \\
5^5 & 6^{a+4} & 7^{b+4} & 1^{c+4} & 2^{d+4} & 3^{e+4} & 4^{f+4} \\
6^6 & 7^{a+5} & 1^{b+5} & 2^{c+5} & 3^{d+5} & 4^{e+5} & 5^{f+5} \\
7^7 & 1^{a+6} & 2^{b+6} & 3^{c+6} & 4^{d+6} & 5^{e+6} & 6^{f+6}
\end{array}$$

5. Odd Magic Squares that also Fulfill the Diagonal Requirement

We set n to be any odd number that will the dimensions of our square and d as the common difference in the guiding formula which is an arithmetic sequence. Then, our squares would be in the form of

$$\begin{array}{ccccc}
1^1 & (1+d)^{1+\delta} & (1+2d)^{1+2\delta} & (1+3d)^{1+3\delta} & \text{etc.} \\
2^2 & (2+d)^{2+\delta} & (2+2d)^{2+2\delta} & (2+3d)^{2+3\delta} & \text{etc.} \\
3^3 & (3+d)^{3+\delta} & (3+2d)^{3+2\delta} & (3+3d)^{3+3\delta} & \text{etc.} \\
4^4 & (4+d)^{4+\delta} & (4+2d)^{4+2\delta} & (4+3d)^{4+3\delta} & \text{etc.} \\
 & & \text{etc.} & &
\end{array}$$

*Image taken from [Eul].

Here, we let δ represent the common difference for the arithmetic progression appearing in the exponent. Now we verify that the conditions are filled. First off, the condition that all the numbers are different in each row and column is satisfied by construction. Now what if we looked at the diagonals? We have the following progression

$$1, 2 + d, 3 + 2d, 4 + 3d, 5 + 4d, \ldots$$

As the common difference is $d + 1$, then as long as $d + 1$ is relatively prime to n, we satisfy our condition. Similarly, by symmetry for the other diagonal, we have that $\delta + 1$ must also be relatively prime to n. In addition, we cannot have $d = \delta$. Note that if n is not prime, we also must have $d \times \delta$ be relatively prime to n.
Here is an example of when $n = 7$. We get the following

I. If $d = 2$ and $\delta = 3$

1^1	3^4	5^7	7^3	2^6	4^2	6^5
2^2	4^5	6^1	1^4	3^7	5^3	7^6
3^3	5^6	7^2	2^5	4^1	6^4	1^7
4^4	6^7	1^3	3^6	5^2	7^5	2^1
5^5	7^1	2^4	4^7	6^3	1^6	3^2
6^6	1^2	3^5	5^1	7^4	2^7	4^3
7^7	2^3	4^6	6^2	1^5	3^1	5^4

II. If $d = 2$ and $\delta = 4$

1^1	3^5	5^2	7^6	2^3	4^7	6^4
2^2	4^6	6^3	1^7	3^4	5^1	7^5
3^3	5^7	7^4	2^1	4^5	6^2	1^6
4^4	6^1	1^5	3^2	5^6	7^3	2^7
5^5	7^2	2^6	4^3	6^7	1^4	3^1
6^6	1^3	3^7	5^4	7^1	2^5	4^2
7^7	2^4	4^1	6^5	1^2	3^6	5^3

III. If $d = 2$ and $\delta = 3$

1^1	3^6	5^4	7^2	2^7	4^5	6^3
2^2	4^7	6^5	1^3	3^1	5^6	7^4
3^3	5^1	7^6	2^4	4^2	6^7	1^5
4^4	6^2	1^7	3^5	5^3	7^1	2^6
5^5	7^3	2^1	4^6	6^4	1^2	3^7
6^6	1^4	3^2	5^7	7^5	2^3	4^1
7^7	2^5	4^3	6^1	1^6	3^4	5^2

IV. If $d = 3$ and $\delta = 4$

1^1	4^5	7^2	3^6	6^5	2^7	5^4
2^2	5^6	1^3	4^7	7^4	3^1	6^5
3^3	6^7	2^4	5^1	1^5	4^2	7^6
4^4	7^1	3^5	6^2	2^6	5^3	1^7
5^5	1^2	4^6	7^3	3^7	6^4	2^1
6^6	2^3	5^7	1^4	4^1	7^5	3^2
7^7	3^4	6^1	2^5	5^2	1^6	4^3

V. If $d = 3$ and $\delta = 5$

1^1	4^6	7^4	3^2	6^7	2^5	5^3
2^2	5^7	1^5	4^3	7^1	3^6	6^4
3^3	6^1	2^6	5^4	1^2	4^7	7^5
4^4	7^2	3^7	6^5	2^3	5^1	1^6
5^5	1^3	4^1	7^6	3^4	6^2	2^7
6^6	2^4	5^2	1^7	4^5	7^3	3^1
7^7	3^5	6^3	2^1	5^6	1^4	4^2

VI. If $d = 4$ and $\delta = 5$

1^1	5^6	2^4	6^2	3^7	7^5	4^3
2^2	6^7	3^5	7^3	4^1	1^6	5^4
3^3	7^1	4^6	1^4	5^2	2^7	6^5
4^4	1^2	5^7	2^5	6^3	3^1	7^6
5^5	2^3	6^1	3^6	7^4	4^2	1^7
6^6	3^4	7^2	4^7	1^5	5^3	2^1
7^7	4^5	1^3	5^1	2^6	6^4	3^2

Now you may be wondering, how does this relate to a magic square? Well, what we can do is replace the base (Latin Numbers) with the numbers $0, 7, 14, 21, 28, 35, 42$ (the multiples of n) and the exponents (Greek Numbers) with the numbers $1, 2, 3, 4, 5, 6, 7$ and then add the numbers in each entry to create our magic square.

For example, if we took number 1 and then converted it into a magic square, we would get

8	25	42	3	20	30	47
16	33	43	11	28	38	6
24	41	2	19	29	46	14
32	49	10	27	37	5	15
40	1	18	35	45	13	23
48	9	26	36	4	21	31
7	17	34	44	12	22	39

by turning the base a into $7a$ (where 7 becomes 0) and for the exponents, it will just map to themselves. We know that this will generate us a working magic circle as we are essentially summing numbers of the form $7a + b$, where the sum of all the a and b in each diagonal,row and column are equal so our bijection will preserve the sums (similar to writing a number in modulo 7). This is also reversible as we can write any number as $7k + l$ for any $l \in \mathbb{Z}$ s.t $l \in [1, 7]$ Another thing to note is that the order in which you assign the values to the numbers does not matter so long as it is unique.

References

[Eul] Leonhard Euler. Recherches sur un nouvelle espèce de quarrés magiques.

Euler Circle
Email address: dieteryang@gmail.com