ON EULER'S FAST CONVERGING SERIES OF π

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In this paper we will discuss Euler's series expansions of π. The initial motivations for finding a new series for π were discussed in [Eul44]. The Leibiniz series, a well-known series for π,

$$\pi = 4\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1},$$

converges extremely slowly where it takes about 10^{50} terms to yield 100 digits. Given the importance of π throughout mathematics, finding an alternative series merits deep investigation.

We should note the Leibniz formula is actually a special case of the following arctan series also known as the Greogry's series

$$\arctan x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}.$$

Now instead of looking at the Leibniz series (i.e. $x = 1$) we instead find smaller series and scale them up to π. For instance we find in the case that $\arctan x = 30^\circ$, we get

$$30^\circ = \arctan \frac{1}{\sqrt{3}} = 6\left[\frac{1}{\sqrt{3}} - \frac{1}{3\cdot 3\sqrt{3}} + \frac{1}{5\cdot 3^2\sqrt{3}} - \frac{1}{7\cdot 3^3\sqrt{3}} + \frac{1}{9\cdot 3^4\sqrt{3}}\right].$$

Scaling up we get,

$$\pi = 6\arctan\frac{1}{\sqrt{3}} = 2\sqrt{3}\left(\frac{1}{1} - \frac{1}{3\cdot 3} + \frac{1}{5\cdot 3^2} - \frac{1}{7\cdot 3^3}\right).$$

This is much better than what we had before. We can get 100 figures of π with 210 terms of calculation. Euler raises the problem that in order to find an approximation of π we need a just as precise approximation for $\sqrt{3}$. We should note that it probably was not a difficult problem to calculate this number for Euler given the recurrences in continued fractions and other tools to generate fast converging series of radicals. However, this is still an intermediate step and it's worth not having to go through that effort if a "better" series exists.

Euler begins this by providing the following shortcut. Note that

$$\arctan\frac{1}{p} = \frac{1}{p} - \frac{1}{3p^3} + \frac{1}{5p^5} - \frac{1}{7p^7} \text{ etc..}$$

Now note the partial sum S where n is even

$$S = \frac{1}{p} - \frac{1}{3p^3} + \frac{1}{5p^5} - \cdots - \frac{1}{(2n-1)p^{2n-1}}.$$

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Then given that partial sum S, the entire sum can be represented as

$$S + \frac{1}{p^{2n+1}}\Big(\frac{1}{(1+p^2)(2n+1)} - \frac{2p^2}{(1+p^2)^2(2n+1)^2} + \frac{2^2(p^4-p^2)}{(1+p^2)^3(2n+1)^3} - \frac{2^3(p^6-4p^4+p^2)}{(1+p^2)^4(2n+1)^4}.\Big)$$

Finally this can be approximated as

$$S + \frac{1}{p^{2n-1}(2n(1+p^2)+p^2-1)},$$

which gets more accurate as we increase the value of n.

Next he makes note of the following arctan identity,

$$\frac{\pi}{4} = \arctan 1 = \arctan\frac{1}{a} + \arctan\frac{1}{b},$$

where a and b are integers such that $\frac{a+b}{ab-1} = 1$.

We now examine the case $a = 2$ and $b = 3$, which gives us the following series

$$\frac{1}{1\cdot 2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \text{ etc.}$$
$$+\frac{1}{1\cdot 3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \text{ etc..}$$

Although we split into two different series, both series converge faster than the one of $\arctan\frac{1}{\sqrt{3}}$. Furthermore, these series do not need the expansion of the square root making the work much easier.

Now it can be found using properties of the shortcut we had used earlier to determine that to find 100 digits of precision, the first series,

$$\frac{1}{1\cdot 2} - \frac{1}{3\cdot 2^3} + \frac{1}{5\cdot 2^5} - \frac{1}{7\cdot 2^7} + \text{ etc.},$$

needs 154 terms calculated which needs to be added to $\frac{1}{2^307\cdot 1543}$

For the second series,

$$+\frac{1}{1\cdot 3} - \frac{1}{3\cdot 3^3} + \frac{1}{5\cdot 3^5} - \frac{1}{7\cdot 3^7} + \text{ etc.},$$

this turns out to be 98 terms, bring the grand total to 252 terms for 100 digits of precision. To get the same result, from $\arctan\frac{1}{\sqrt{3}}$ we would have also needed over 200 terms. Again, given the benefits we outlined above, this is still a better series.

In, [Eul98], Euler uses a slightly different approach. He continues to use arctan identities, but uses the power series for the following polynomial as opposed to the Gregory series. Euler begins with the binomial

$$x^4 + 4 = (2 + 2x + x^2)(2 - 2x + x^2).$$

Next we note that

$$f(x) = \int \frac{x^2+2x+2}{x^4+4}dx = \int \frac{1}{x^2-2x+2}dx = \arctan\frac{x}{2-x}$$

Next we calculate we note the following antiderivatives,

$$f(1) = \frac{\pi}{4}, f\left(\frac{1}{2}\right) = \arctan\frac{1}{3}, f\left(\frac{1}{4}\right) = \arctan\frac{1}{7}.$$

Using these antiderivatives, we'll generate the series for π from the following arctan,

$$8\arctan\frac{1}{3} + 4\arctan\frac{1}{7} = \arctan 1 = \pi. \tag{0.1}$$

Now let's define the following integrals

$$a = \int \frac{dx}{x^4+4}; b = \int \frac{xdx}{x^4+4}; c = \int \frac{x^2dx}{x^4+4}.$$

Thus

$$f(x) = 2a + 2b + c \tag{0.2}$$

The integrand of a can be expressed as

$$\frac{1}{x^4+4} = \frac{\frac{1}{4}}{1+\frac{x^4}{4}} = \frac{1}{4}\sum_{i=0}^{\infty}\left(-\frac{x^4}{4}\right)^i.$$

We can obtain similar series for the integrands of b and c by multiplying x and x^2 respectively.

Now we cane obtain a, b, and c as the following series from integrating the terms in the power series:

$$a = \frac{x}{4}\left[1 - \frac{1}{5}\frac{x^4}{4} + \frac{1}{9}\left(\frac{x^4}{4}\right)^2 - \frac{1}{3}\left(\frac{x^4}{4}\right)^3 + \dots\right]$$

$$b = \frac{x^2}{8}\left[1 - \frac{1}{3}\frac{x^4}{4} + \frac{1}{5}\left(\frac{x^4}{4}\right)^2 - \frac{1}{7}\left(\frac{x^4}{4}\right)^3 + \dots\right]$$

$$c = \frac{x^3}{4}\left[\frac{1}{3} - \frac{1}{3}\frac{x^4}{4} + \frac{1}{11}\left(\frac{x^4}{4}\right)^2 - \frac{1}{15}\left(\frac{x^4}{4}\right)^3 + \dots\right].$$

For our first case $f(1) = \frac{\pi}{4}$. We we get the following values for a, b and c.

$$a = \frac{1}{4}\left(1 - \frac{1}{5}\cdot\frac{1}{4} + \frac{1}{9}\cdot\frac{1}{4^2} - \frac{1}{13}\cdot\frac{1}{4^3} + \frac{1}{17}\cdot\frac{1}{4^4} - \text{ etc.}\right)$$
$$b = \frac{1}{8}\left(1 - \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4^2} - \frac{1}{7}\cdot\frac{1}{4^3} + \frac{1}{9}\cdot\frac{1}{4^4} - \text{ etc.}\right)$$
$$c = \frac{1}{4}\left(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{4} + \frac{1}{11}\cdot\frac{1}{4^2} - \frac{1}{13}\cdot\frac{1}{4^3} + \frac{1}{19}\cdot\frac{1}{4^4} - \text{ etc.}\right)$$

Using the formula (0.2) we get

$$\pi = 2\left(1 - \frac{1}{5}\cdot\frac{1}{4} + \frac{1}{9}\cdot\frac{1}{4^2} - \frac{1}{13}\cdot\frac{1}{4^3} + \frac{1}{17}\cdot\frac{1}{4^4} - \dots\right) + 1\left(1 - \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{5}\cdot\frac{1}{4^2} - \frac{1}{7}\cdot\frac{1}{4^3} + \frac{1}{9}\cdot\frac{1}{4^4} - \dots\right) + 1\left(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{4} + \frac{1}{11}\cdot\frac{1}{4^2} - \frac{1}{15}\cdot\frac{1}{4^3} + \frac{1}{19}\cdot\frac{1}{4^4} - \dots\right). \tag{0.3}$$

This series of π performs much better compared to Leibniz given that each term in the series decreases by a factor of 4. However, although it may seem we've completed our job, we can do even better by expanding the rest of the terms. For our next case of $x = \frac{1}{2}$ we get,

$$a = \frac{1}{8}\left(1 - \frac{1}{5}\cdot\frac{1}{64} + \frac{1}{9}\cdot\frac{1}{64^2} - \frac{1}{13}\cdot\frac{1}{64^3} + \text{ etc.}\right)$$
$$b = \frac{1}{32}\left(1 - \frac{1}{3}\cdot\frac{1}{64} + \frac{1}{5}\cdot\frac{1}{64^2} - \frac{1}{7}\cdot\frac{1}{64^3} + \text{ etc.}\right)$$
$$c = \frac{1}{32}\left(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{64} + \frac{1}{11}\cdot\frac{1}{64^2} - \frac{1}{15}\cdot\frac{1}{64^3} + \text{ etc.}\right),$$

which culminates to

$$\arctan\frac{1}{3} = \frac{1}{4}\left(1 - \frac{1}{5}\cdot\frac{1}{64} + \frac{1}{9}\cdot\frac{1}{64^2} - \frac{1}{13}\cdot\frac{1}{64^3} + \text{ etc.}\right) + \frac{1}{16}\left(1 - \frac{1}{3}\cdot\frac{1}{64} + \frac{1}{5}\cdot\frac{1}{64^2} - \frac{1}{7}\cdot\frac{1}{64^3} + \text{ etc.}\right) + \frac{1}{32}\left(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{64} + \frac{1}{11}\cdot\frac{1}{64^2} - \frac{1}{15}\cdot\frac{1}{64^3} + \text{ etc.}\right). \tag{0.4}$$

For our final case of $x = \frac{1}{4}$ we get,

$$a = \frac{1}{16}(1 - \frac{1}{5}\cdot\frac{1}{1024} + \frac{1}{9}\cdot\frac{1}{1024^2} - \frac{1}{13}\cdot\frac{1}{1024^3} + \text{ etc.})$$
$$b = \frac{1}{128}(1 - \frac{1}{3}\cdot\frac{1}{1024} + \frac{1}{5}\cdot\frac{1}{1024^2} - \frac{1}{7}\cdot\frac{1}{1024^3} + \text{ etc.})$$
$$c = \frac{1}{256}(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{1024} + \frac{1}{11}\cdot\frac{1}{1024^2} - \frac{1}{15}\cdot\frac{1}{1024^3} + \text{ etc.})$$

which similarly culminates to,

$$\arctan\frac{1}{7} = \frac{1}{8}(1 - \frac{1}{5}\cdot\frac{1}{1024} + \frac{1}{9}\cdot\frac{1}{1024^2} - \frac{1}{13}\cdot\frac{1}{1024^3} + \text{ etc.}) + \frac{1}{64}(1 - \frac{1}{3}\cdot\frac{1}{1024} + \frac{1}{5}\cdot\frac{1}{1024^2} - \frac{1}{7}\cdot\frac{1}{1024^3} + \text{ etc.}) + \frac{1}{256}(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{1024} + \frac{1}{11}\cdot\frac{1}{1024^2} - \frac{1}{15}\cdot\frac{1}{1024^3} + \text{ etc.}). \tag{0.5}$$

Finally we'll substitute (0.1) with (0.4) and (0.5) to get our final series of

$$\begin{aligned}
\pi = {} & 2(1 - \frac{1}{5}\cdot\frac{1}{64} + \frac{1}{9}\cdot\frac{1}{64^2} - \frac{1}{13}\cdot\frac{1}{64^3} + \text{ etc.}) \\
& + \frac{1}{2}(1 - \frac{1}{3}\cdot\frac{1}{64} + \frac{1}{5}\cdot\frac{1}{64^2} - \frac{1}{7}\cdot\frac{1}{64^3} + \text{ etc.}) \\
& + \frac{1}{4}(\frac{1}{3} - \frac{1}{7}\cdot\frac{1}{64} + \frac{1}{11}\cdot\frac{1}{64^2} - \frac{1}{15}\cdot\frac{1}{64^3} + \text{ etc.}) \\
& + \frac{1}{2}(1 - \frac{1}{5}\cdot\frac{1}{1024} + \frac{1}{9}\cdot\frac{1}{1024^2} - \frac{1}{13}\cdot\frac{1}{1024^3} + \text{ etc.}) \\
& + \frac{1}{16}(1 - \frac{1}{3}\cdot\frac{1}{1024} + \frac{1}{5}\cdot\frac{1}{1024^2} - \frac{1}{7}\cdot\frac{1}{1024^3} + \text{ etc.}) \\
& + \frac{1}{64}(1 - \frac{1}{7}\cdot\frac{1}{1024} + \frac{1}{11}\cdot\frac{1}{1024^2} - \frac{1}{15}\cdot\frac{1}{1024^3} + \text{ etc.})
\end{aligned}$$

This performs even better than the new series that we had found earlier with the expression given that our terms are now decreasing by factors of 64 and 1024. Furthermore, Euler notes the utility of this series given that the terms expressible by an ever decreasing power of 2, making particularly efficient to write out π in binary.

References

[Eul44] Leonhard Euler. De variis modis circuli quadraturam numeris proxime exprimendi. *Commentarii academiae scientiarum Petropolitanae*, 9:222–236, (1737) 1744.

[Eul98] Leonhard Euler. De novo genere serierum rationalium et valde convergentium quibus ratio peripheriae ad diametrum exprimi potest. *Nova acta academiae scientiarum Petropolitinae*, 11:150–154, 169, (1793) 1798.