E52: Euler's Paper on Elliptic Integrals

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Abstract

In this paper, I review Leonhard Euler's findings regarding the rectification of ellipses through elliptic integrals and go through the steps Euler took to find a second-degree differential equation for the length of the arc of an ellipse.

Introduction

To preface, rectification is the representation of the length of a curve through the length of a corresponding straight line.

Before Euler's paper on Elliptic Integrals [1], mathmeticians had only ever solved the rectification of circles and parabolas. However, in his paper from 1741, Euler went into detail about how he completed the rectification of ellipses.

General Rectification of Ellipses

The biggest issue Euler had in this process was that the rectification of these different ellipses did not have any relationships with one-another. As labeled in Figure A, he used line segments $\overline{AP}$, $\overline{AC}$, $\overline{PM}$, and $\overline{CD}$ to create a differential equation relating them to arc AM.

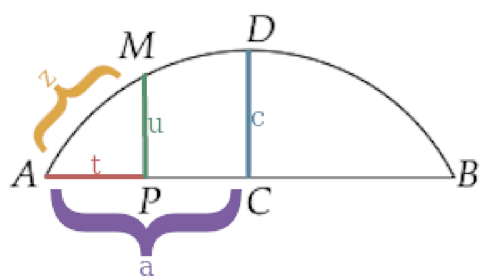

Figure A: A half-ellipse (Modified from E52 [1])

Euler began by naming the sides connecting the different points: he set $\overline{AP} = t$, $\overline{AC} = a$, $\overline{PM} = u$, $\overline{CD} = c$, and arc $AM = z$, where t, a, u, and z are variable lengths while c is constant. First, Euler saw that u was proportional to c, and that the maximum u could be was c. He also noticed that the ratio between u and t was proportional to the ratio between c and a. This was the equation he found:

$$u = \frac{c}{a}\sqrt{2at - t^2}.$$

Now, he set $t = ax$, where x is some variable:

$$u = \frac{c}{a}\sqrt{2a^2x - a^2x^2} = c\sqrt{2x - x^2}.$$

Then, he found two derivatives:

$$dt = a\,dx \qquad \texttt{and} \qquad du = \frac{c\,dx - cx\,dx}{\sqrt{2x - x^2}}.$$

Now, from Figure A, we can say that

$$dz = \sqrt{du^2 + dt^2} = \sqrt{du^2 + a^2\,dx^2}.$$

Euler then came to the following conclusion:

$$dz = dx\frac{\sqrt{2a^2x - a^2x^2 + c^2 - 2c^2x + c^2x^2}}{\sqrt{2x - x^2}}.$$

He proceeded by setting $a^2 - c^2 = b^2$ in the above equation:

$$\frac{dz}{dx} = \frac{\sqrt{c^2 + b^2(2x - x^2)}}{\sqrt{2x - x^2}}.$$

Here, it is clear also that

$$z = \int \frac{\sqrt{c^2 + b^2(2x - x^2)}}{\sqrt{2x - x^2}}\,dx.$$

Then he took the derivative again, this time with respect to b:

$$\frac{dz^2}{dx^2} = \frac{b\sqrt{2x - x^2}}{\sqrt{c^2 + b^2(2x - x^2)}}\,db,$$

or equivalently:

$$dz = \int \frac{b\sqrt{2x - x^2}}{\sqrt{c^2 + b^2(2x - x^2)}}\,db\,dx.$$

From here, he created the following equation:

$$dz = \frac{\sqrt{c^2 + b^2(2x - x^2)}}{\sqrt{2x - x^2}}\,dx + \int \frac{b\sqrt{2x - x^2}}{\sqrt{c^2 + b^2(2x - x^2)}}\,dx\,db,$$

where in the integral, b is a constant. This equation follows because Euler took the derivative dz as a function of x and b (i.e. the derivative of z with respect to x plus the derivative of z with respect to b). After this, he equated the following:

$$R = \frac{dz}{db} - \frac{dx}{db}\frac{\sqrt{c^2 + b^2(2x - x^2)}}{\sqrt{2x - x^2}},$$

for some R. And so, the following is true as well:

$$R = \int \frac{b\sqrt{2x - x^2}}{\sqrt{c^2 + b^2(2x - x^2)}}\,db,$$

in which b is again the variable. He followed the same sequence of steps for R as for z to get

$$dR = \frac{b\sqrt{2x - x^2}}{\sqrt{c^2 + b^2(2x - x^2)}}\,dx + \int \frac{c^2\sqrt{2x - x^2}}{[c^2 + b^2(2x - x^2)]^{\frac{3}{2}}}\,db\,dx.$$

Now again, Euler made another variable, S, and equated the following:

$$S = \frac{dR}{db} - \frac{dx}{db}\frac{b\sqrt{2x - x^2}}{\sqrt{c^2 + b^2(2x - x^2)}} = \int \frac{c^2\sqrt{2x - x^2}}{[c^2 + b^2(2x - x^2)]^{\frac{3}{2}}}\,dx.$$

Now, he looked to see if S has a relationship with z and/or r. To begin, he made a $Q = S + \alpha R + \beta z$ such that α and β are independent from x and z and that $Q = 0$ when $x = 0$. Clearly, taking the derivative of both sides results in $dQ = dS + \alpha dR + \beta dz$. When b is again a constant, one can find the derivatives of S, R, and z as such:

$$\frac{dS}{dx} = \frac{c^2\sqrt{2x-x^2}}{[c^2+b^2(2x-x^2)]^{\frac{3}{2}}}, \quad \frac{dR}{dx} = \frac{b\sqrt{2x-x^2}}{\sqrt{c^2+b^2(2x-x^2)}}, \quad \texttt{and} \quad \frac{dz}{dx} = \frac{\sqrt{c^2+b^2(2x-x^2)}}{\sqrt{2x-x^2}}.$$

Now we can simply plug in these derivatives to get

$$\frac{dQ}{dx} = \frac{\left[c^2(2x-x^2)+abc^2(2x-x^2)+\alpha b^3(2x-x^2)^2+\beta c^4+2\beta b^2c^2(2x-x^2)+\beta b^4(2x-x^2)^2\right]}{[c^2+b^2(2x-x^2)]^{\frac{3}{2}}\sqrt{2x-x^2}}.$$

Euler then had the idea to add in two new variables, γ and δ to simplify dQ, as follows:

$$Q = \frac{(\gamma x+\delta)\sqrt{2x-x^2}}{\sqrt{c^2+b^2(2x-x^2)}} \quad\longrightarrow\quad \frac{dQ}{dx} = \frac{\gamma c^2(2x-x^2)+\gamma b^2(2x-x^2)^2+\gamma c^2x+\delta c^2-\gamma c^2x^2-\delta c^2x}{[c^2+b^2(2x-x^2)]^{\frac{3}{2}}\sqrt{2x-x^2}}.$$

It is possible now to combine like terms and find out the values of α, β, γ, and δ, just like Euler did:

$$\alpha = \frac{1}{b}, \qquad \beta = -\frac{1}{b^2+c^2}, \qquad \gamma = \frac{c^2}{b^2+c^2}, \qquad \delta = -\frac{c^2}{b^2+c^2}.$$

Knowing these values, we can now equate our two equations of Q:

$$Q = S + \alpha R + \beta z = S + \frac{R}{b} - \frac{z}{b^2+c^2}$$
$$Q = \frac{(\gamma x+\delta)\sqrt{2x-x^2}}{\sqrt{c^2+b^2(2x-x^2)}} = \frac{c^2(x-1)\sqrt{2x-x^2}}{(b^2+c^2)\sqrt{c^2+b^2(2x-x^2)}}.$$

After some more intense algebraic manipulation, Euler created Q in terms of a, c, and t:

$$Q = \frac{c^2(t-a)\sqrt{2at-t^2}}{a^3\sqrt{a^2c^2+(a^2-c^2)(2at-t^2)}}.$$

At this point, Euler made the complicated but trivial substitutions and manipulations in Q to get that

$$\frac{z}{b^2+c^2} = \frac{c^2(1-x)\sqrt{2x-x^2}}{(b^2+c^2)\sqrt{c^2+b^2(2x-x^2)}} - \frac{dx}{db}\frac{1}{b}\sqrt{\frac{c^2+b^2(2x-x^2)}{2x-x^2}} - \frac{dx}{db}2b\sqrt{\frac{2x-x^2}{c^2+b^2(2x-x^2)}}$$
$$+ \frac{dx^2}{db^2}\frac{c(1-x)}{(2x-x^2)^{\frac{3}{2}}\sqrt{c^2+b^2(2x-x^2)}} + \frac{dz}{b\,db} + \frac{1}{db}\frac{dz^2}{db^2} - \frac{1}{db}\frac{dx^2}{db^2}\sqrt{\frac{c^2+b^2(2x-x^2)}{2x-x^2}},$$

a second-degree differential equation for the arc AM.

As an example of how this differential equation is useful, I will go over Problem 1 from Euler's paper.

Problem 1

Problem 1 concerns different quarters of ellipses. Euler attempted to find a curve of which the height at different points of it is the length of the corresponding curve of an ellipse. So extending or shortening segment $\overline{PM}$ along curve EMN to get the length of some semi-ellipse AF, where P, F, and M are not necessarily in the center of axis AQ (see Figure B).

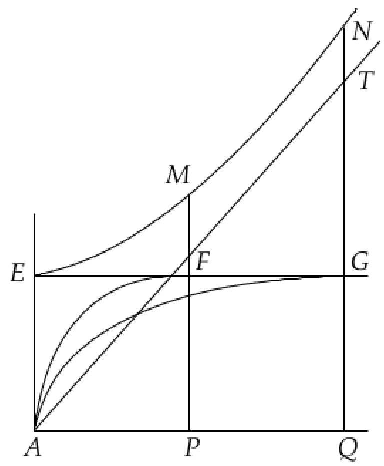

Figure B: Diagram for Problem 1 ([1])

Using the same definitions for the variables as from before, he began by setting $x = 1$ or $t = a$ (resulting in z being arc AF). He got that

$$\frac{z}{b^2 + c^2} = \frac{dz}{b\,db} + \frac{1}{db}\frac{dz^2}{db^2}.$$

Since b^2 is now $t^2 - c^2$ and $b\,db = a\,da = t\,dt$, db is $\frac{t\,dt}{\sqrt{t^2-c^2}}$ and $db^2 = -\frac{c^2\,dt^2}{(t^2-c^2)^{\frac{3}{2}}}$ (setting dt as a constant). Knowing this, we can next calculate $\frac{dz^2}{db^2}$:

$$\frac{dz^2}{db^2} = \frac{dz^2}{dt}\frac{\sqrt{t^2 - c^2}}{t} + \frac{c^2\,dz}{t^2\sqrt{t^2 - c^2}}.$$

Taking the integral twice, we get

$$\frac{z}{t^2} = \frac{dz}{t\,dt} + \frac{dz^2}{dt^2}\frac{(t^2 - c^2)}{t^2} + \frac{dz}{dt}\frac{c^2}{t^3}.$$

This is equivalent to

$$tz\,dt^2 = (t^2 + c^2)\,dt\,dz + t\,dz^2(t^2 - c^2).$$

Again, we have found another relationship between z, t, and c, showing the rectification of some curve AF intersected by $\overline{PM}$ which is intersected by curve EMN. This is true in general because AF is a quarter of an ellipse and simply changing values like a or t only creates a different quarter which we can rectify the same way.

Euler also chose to solve two more problems, which I will outline to show the uses of the differential equation.

Problem 2

The next task Euler chose to complete was Problem 2, in which he attempted to find some curve at which many different curves of ellipses would meet, and be the same length. In other words, he found some curve $BONMC$ which curves AOF, ANG, and AMH (of ellipses) would intersect, where AO, AN, and AM have the same length (as seen in Figure C).

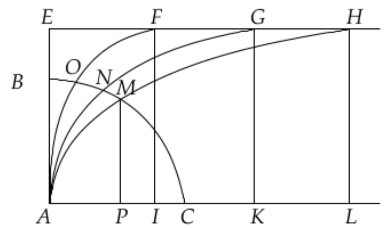

Figure C: Diagram for Problem 2 ([1])

Problem 3

Lastly, in Problem 3, Euler found a curve that would section off multiple ellipses with a universal center, to be the same length. So, quarter-ellipses AMF, ANG, and AOH all have center C and are intersected by some unknown curve ONM (see Figure D).

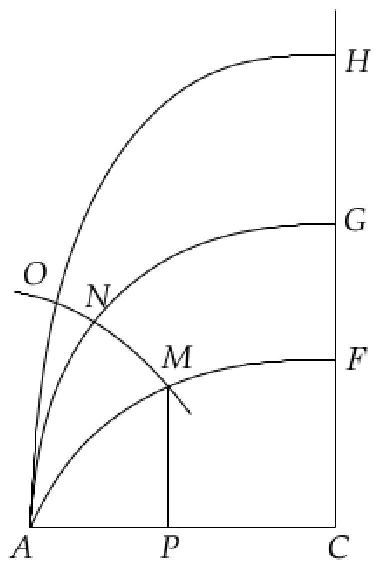

Figure D: Diagram for Problem 3 (Adapted from E52 [1])

References

[1] Euler Leonhard. E52: Lösung der probleme, die die rektifikation der ellipse erfordern, 1741. Translated by Alexander Alycock.