A Spin to the Calculation of Sines

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Abstract

Swiss mathematician and physicist, Leonhard Euler, used the Maclaurin series, or power series to find his formula, Euler's formula, which is $e^{ix} = \cos(x) + i \cdot \sin(x)$. This famous formula is used to establish the relationship between trigonometric functions and the complex exponential function. The complex exponential function has properties that make it a versatile tool in various subjects, including mathematics, finance, engineering, and computer science.

1 Introduction

Leonhard Euler wrote a paper "Subsidium calculi sinuum," or "A contribution to the calculations of sines" where he focused on the sine function and how the trigonometry relates to the exponential function.

The complex exponential function is a fundamental concept that extends the traditional exponential function to the complex plane. It is represented as e^z where e is Euler's number, and z is a complex number of the form $a + bi$, where a and b are real numbers and i is a imaginary unit.

It is crucial to note that Euler's Formula and DeMoiré's Theorem are very similar in what they both state. That $(\cos(\phi) + i \cdot \sin(\phi))^n = \cos(n\phi) + i \cdot \sin(n\phi)$. However, DeMoiré's theorem was only proved for all $n \in N$. For half a century after DeMoiré's Theorem, no one was able to prove that the formula would hold true for complex numbers.

2 Deriving DeMoiré's Theorem

First, Euler derives DeMoiré's Theorem in Lemma 1, by demonstrating the property with two angles, ϕ and α. He continued to simplify the following expression:

$(\cos(\Phi) + i \cdot \sin(\Phi))(\cos(\alpha) + i \cdot \sin(\alpha))$

He multiplied the binomials by distributing each term:
$= (\cos(\Phi)\cos(\alpha)) + (\cos(\Phi)\ i\sin(\alpha)) + (\cos(\alpha)\ i\sin(\phi)) + (i\sin(\Phi)\ i\sin(\alpha))$

Since $i = \sqrt{-1}$, therefore, $i^2 = -1$, when simplified, the above becomes:
$= (\cos(\Phi)\cos(\alpha)) + (\cos(\Phi)\ i\sin(\alpha)) + (\cos(\alpha)\ i\sin(\phi)) - (\sin(\Phi)\sin(\alpha))$

Now, once the i is factored out, the resulting expression is:
$= (\cos(\Phi)\cos(\alpha)) + i(\sin(\Phi)\cos(\alpha)) + (\cos(\phi)\ i\sin(\alpha)) - (\sin(\Phi)\ \sin(\alpha))$

The terms were regrouped to form a familiar identity
$= (\cos(\Phi)\cos(\alpha)) - (\sin(\Phi)\ \sin(\alpha)) + i(\sin(\Phi)\cos(\alpha)) + (\cos(\Phi)\ \sin(\alpha))$

Since $\cos(\Phi + \alpha) = (\cos(\Phi)\cos(\alpha)) - (\sin(\Phi)\ \sin(\alpha))$, the expression can be rewritten as:
$\cos(\Phi + \alpha) + i(\sin(\Phi)\cos(\alpha)) + (\cos(\Phi)\ \sin(\alpha))$

Another familiar identity is the sum of sines. In other words, $\sin(\Phi + \alpha) = (\sin(\Phi)\cos(\alpha)) + (\cos(\Phi)\ \sin(\alpha))$. If we substitute this property into the equation, we get:

$\cos(\Phi+\alpha)+i(\sin(\Phi+\alpha))$

Following this, Euler analyzed the case that $\alpha=\Phi$. The equation becomes:
$((\cos(\Phi)+i(\sin(\Phi))^2=\cos(2\Phi)+i\sin(2\Phi)$

If $\cos(2\Phi)+i\sin(2\Phi)$ is multiplied by $\cos(\Phi)+i\sin(\Phi)$, the new equation is:
$((\cos(\Phi)+i(\sin(\Phi))^2=(\cos(2\Phi)+i\sin(2\Phi))(\cos(\Phi)+i\sin(\Phi))=\cos(3\Phi)+i\sin(3\Phi)$
Euler sets $(\cos(\Phi))+i(\sin(\Phi))=u$ and v is the conjugate of u. Therefore, $(\cos(\Phi))-i(\sin(\Phi))=v$. Based on Lemma 1:

$u^n=(\cos(n\Phi))+i(\sin(n\Phi))$
$v^n=(\cos(n\Phi))-i(\sin(n\Phi))$

The sum of u^n and v^n is:
$u^n+v^n=\cos(n\Phi)+i\sin(n\Phi)+\cos(n\Phi)-i(\sin(n\Phi))=2\cos(n\Phi)$

The difference of v^n from u^n is:
$u^n-v^n=\cos(n\Phi)+i\sin(n\Phi)-\cos(n\Phi)+i(\sin(n\Phi))=2i\sin(n\Phi)$

Furthermore, it can also be agreed that $uv=1$ since:
$uv=(\cos(\Phi)+i\sin(\Phi))(\cos(\Phi)+i\sin(\Phi))=\cos^2(\Phi)-i\sin(\Phi)\cos(\Phi)+i\sin(\Phi)\cos(\Phi)-i^2\sin^2(\Phi)=\cos^2(\Phi)+\sin^2(\Phi)$
Due to the Pythagorean Identity, $\cos^2(\Phi)+\sin^2(\Phi)=1$. Therefore, $uv=1$

Next, Euler wanted to convert the power of the cosine of some angle, Φ, so that no two or more cosines are multiplied by each other. To do this, he took $\cos(\Phi)^n$. Let $u=\cos(\Phi)+i\sin(\Phi)$ and $v=\cos(\Phi)i\sin(\Phi)$. Therefore $u+v=2\cos(\Phi)$, so $\cos(\Phi)=\frac{(u+v)}{2}$, and $\cos^n(\Phi)=\frac{(u+v)^n}{2^n}$.
$=2^n\cos^n(\Phi)=(u+v)^n$
Next, the right hand side of the equation undergoes binomial expansion. That expands to:
$=(u+v)^n=\binom{n}{0}u^nv^0+\binom{n}{1}u^{n-1}v^1+\binom{n}{2}u^{n-2}v^2+\binom{n}{3}u^{n-3}v^3+\ldots=u^n+nu^{n-1}v+\frac{n(n-1)}{1\cdot 2}u^{n-2}v^2+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}u^{n-3}v^3+...$
When the order of of u and v are switched, the left side remains the same, and the order of the terms on the right side are switched as well. The new equation becomes:
$=2^{n+1}\cos^n(\Phi)=u^n+v^n+n(u^{n-2}+v^{n-2})uv+\frac{n(n-1)}{1\cdot 2}(u^{n-4}+v^{n-4})u^{n-2}v^2+...$

Seeing that as $uv=1$, if both sides are divided by 2, the equation becomes:
$=2^n\cos^n(\Phi)=\frac{1}{2}(u^n+v^n)+\frac{n}{2}(u^{n-2}+v^{n-2})uv+\frac{n(n-1)}{1\cdot 2\cdot 2}(u^{n-4}+v^{n-4})u^{n-2}v^2+...$

Substituting the occurrences of u^n+v^n with $2\cos(n\Phi)$ gives the equation:
$=2^n\cos^n(\Phi)=\frac{1}{2}(2\cos(n\Phi))+n(\cos((n-2)\Phi))+\frac{n(n-1)}{1\cdot 2}(\cos((n-4)\Phi))+...$
It is important to observe that:
$\cos((n-m)\Phi)=\cos((m-n)\Phi)$

3 Euler's Complex Exponential Function

3.1 Proving the Complex Exponential Function Using a Taylor Series

The formula for Euler's Complex Exponential Function states: $e^{i\Phi}=\cos(\Phi)+i\sin(\Phi)$. Where e is the base of the natural logarithm. To prove this formula, we need to find the Taylor series expansions for the exponential , cosine, and sine functions.
A Taylor series is usually used to approximate functions with polynomials. Generally, the formula for a Taylor series expansion for a function $f(x)$ that is centered at $(a,f(a))$ is $f(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\frac{f'''(a)}{3!}(x-a)^3+\ldots$

For the exponential function, the Taylor series expansion is: $e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}=e^0+e^0x+\frac{e^0}{2!}x^2+\frac{e^0}{3!}x^3+\cdots=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

The Taylor series expansion of the $\cos(x)$ function centered at $a = 0$, and with a derivative evaluated at $x = 0$ is: $\cos(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k}}{(2k)!} = \cos(0) - \sin(0)x - \frac{\cos(0)}{2!}x^2 + \frac{\sin(0)}{3!}x^3 + \cdots = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$ Additionally, when the $\sin(x)$ function that is centered at $a = 0$, and with a derivative evaluated at $x = 0$ goes through a Taylor series expansion, the result is: $\sin(x) = \sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)!} = \sin(0) + \cos(0)x - \frac{\sin(0)}{2!}x^2 - \frac{\cos(0)}{3!}x^3 + \cdots = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$

Now that the Taylor expansions of the required functions are known, the substitution of the functions with the expansions can be made.

$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ldots$

After the powers of i are simplified and the common terms are factored out of the equation, the resulting equation is:

$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \ldots$

When the imaginary and real parts of the series are separated the equation is:

$e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\right)$ When adding the expansions of the $\cos(x)$ and $i\sin(x)$ functions, the resulting equation is:

$\cos(x) + i\sin(x) = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} - \ldots$

3.2 Justifying the Complex Exponential Function Using Derivatives

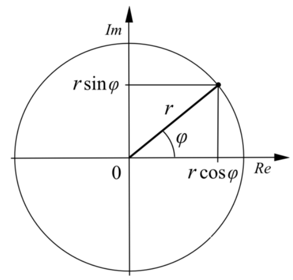
Let the function, $f(x) = \cos(x) + i\sin(x)$. After taking the derivative of this function, the result is: $f'(x) = -i\sin(x) + \cos(x)$. As we can see, $\frac{f'(x)}{f(x)} = i$. In other words, this function has the property that it's derivative is i times the function itself. This means $\frac{dg}{dx} = ig$. To solve this differential equation, a convenient method is through the separation of variables. $\frac{1}{g}dg = idx$. Integrating both sides gives us:
$\int \frac{1}{g}dg = \int idx$
$= \ln|g| = ix$
$= ln|g| = ix$
$= |g| = e^{ix} + C = e^C e^{ix}$
If we let e^C be equal to some variable C_2, then:
$|g| = C_2 e^{ix}$
Therefore, $g = C_3 e^{ix}$. Now, we must find the value of constant C_3 that will make $f(x) = g(x)$. Let $x = 0$. We get the following:
$= f(x) = \cos(0) + i\sin(0) = 1$
$= g(x) = C_3 e^{i0} = C_3$
Hence, these two functions are equal when $C_3 = 1$
Therefore, $\cos(x) + i\sin(x) = e^{ix}$.

3.3 Exponential Form

Additionally, we can also write that a complex number $z = re^{i\theta}$ where r is the radius of the circle. This representation describes complex numbers as exponential functions by providing information about both their magnitude, r, and their phase, θ.

3.4 Geometrical Representation

Euler's formula provides a means to express any given complex number z as e^{ix} , positioning it on a unit circle characterized by its real component $\cos(x)$ and imaginary component $i\sin(x)$. Within this framework, operations like determining the roots of unity can be visualized as rotational movements along the circumference of the unit circle. The point e^{ix} traces out a unit circle in the complex plane as x varies.



3.5 Euler's Famous Identity

Also known as the most beautiful equation in mathematics, Euler's famous identity stems from the special case of $\Phi = \pi$. Since $e^{i\pi} = -1$, by adding 1 to both sides, the resulting equation is $e^{i\pi} + 1 = 0$. This important equation allows mathematicians to relate the four numbers, e, π, i, and 0. Using Taylor series, we proved this formula by breaking down the formula into which infinitely many terms.

4 Roots of Unity

Roots of unity are complex numbers that, when raised to certain powers, equal 1. These special numbers lie on the unit circle in the complex plane, where the unit circle is a circle centered at the origin with a radius of 1. The nth roots of unity are equally spaced around the unit circle, with each root separated by an angle of $\frac{2\pi}{n}$ radians. The formal definition of the nth root of unity is: for any positive integer n, the nth roots of unity are the complex solutions to the equation $x^n = 1$ and there are n solutions to the equation. Euler's formula can be used to find the nth roots of unity for any positive integer n. We can say $U_n = \left\{ e^{\frac{2k\pi i}{n}} \,\middle|\, k \in \{1, 2, \ldots, n\} \right\}$ where U_n represents the set of all nth roots of unity. These roots of unity are often written in polar form as $e^{\frac{2k\pi i}{n}}$, where k is the set of natural numbers until n.

References

E246 – Subsidium calculi sinuum. (n.d.). Retrieved August 20, 2023, from http://eulerarchive.maa.org/backup/E246.html

Euler, L. (1760). Subsidium calculi sinuum. https://scholarlycommons.pacific.edu/euler-works/246

Larson, Caleb (2017) "An Appreciation of Euler's Formula," Rose-Hulman Undergraduate Mathematics Journal: Vol. 18 : Iss. 1 , Article 17.