Expository Paper on E591: On Establishing a Relationship Between Three or More Quantities

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ABSTRACT
Leonhard Euler was a mathematician in the 18th century who advanced many mathematical fields. One of his papers was E591: "On Establishing a Relationship Between Three or More Quantities" which explores linear relationships between different types of numbers. We will go over the paper and explore it from multiple angles.

1. Introduction

Leonhard Euler was a mathematician in the 18th century and contributed many advancements to various mathematical fields. One of these fields was linear equations. Euler wrote a paper in 1775 (which was published posthumously, in 1785) titled "De relatione inter ternas pluresve quantitates instituenda", translating to "On Establishing a Relationship Among Three or More Quantities", with Enestrom number E591.

This paper goes into detail about relating three or more numbers using linear equations. It starts with Diophantine-esque equations, where there are two numbers A and B, resulting in the equation $aA = bB$ where a and b are integers.

2. Playing Around

Similarly, this can be extended to any number of numbers A, B, C, etc.

For reference, we will use 3 numbers so that $aA = bB = cC$. As an example we will use $A = 49$, $B = 59$, and $C = 75$. Now divide both equations by $A = 49$, and separate this into integral and fractional parts: $a + b + c = d$ and $(10b + 26c)/49 = -d$. d is a new quantity.

We can continually do this over and over again, with new quantities. After continuing, we get $d/2 = -f$ and the chain ends.

Applying the results appropriately, we get the equations

$$c = 3f - 5e$$

,

$$b = 13e + 2f$$

,

$$a = -8e - 7f$$

The original equation had multiple solutions, and they will have the form $-(8e+7f)\cdot 49+(13e+2f)\cdot 59+(3f-5e)\cdot 75 = 0$.

This chain of equations works somewhat like repeatedly adding a constant k to a number n and eventually finding the residue mod M to be 0.

We can also see that for an irrational number, the final linear equation involving the solutions will be infinitely long, which is why we cannot compare these types of numbers using this method.

3. Taking it Further

We can see that ANY rational number should eventually yield integer solutions. However, any irrational or transcendental number will not yield solutions, as it will simply go on forever.

However, we can still approximate the solutions by approximating the irrational number; the degree of accuracy depends on the degree of rounding.

In order to convert irrational numbers to a form that can be compared using these linear equations, we simply multiply by whatever place we want to round to, and then round to the nearest integer. So rounding to millionths, $\sqrt{2} = 1.414214...$ is converted to 1414214. We can then set up a relation involving $\sqrt{3} and \pi$: $1414214a + 1732051b + 3141593c = 0$. Without going into the full extent (because it would take a long time to complete), this can give us a somewhat approximate way of relating $\sqrt{2}, \sqrt{3}, and \pi$.

4. Infinite Series

We can also represent infinite series using these linear combinations. For example, take the series

$$\sum_{n=1} \infty \frac{1}{n^3} = 1 + \frac{1}{2^3} + \frac{1}{3^3} \cdots$$

- as it is the sum of reciprocal cubes, it should involve

$$\ln 2 = \sum_{n=2}^{\infty} \frac{1}{(-1)^n(n-1)}$$

And it is also possible that it involves $\frac{\pi^2}{6}$, because it contains the sum of reciprocal squares.

Suppose

$$A = \sum_{n=1}^{\infty} \frac{(-1)^(n+1)}{n^3}$$

. We can set up equations so that

$$aA + b \ln 2^3 + c \ln 2 \frac{\pi^2}{6} = 0$$

. Based on the approximate decimal expansions of $\frac{\pi^2}{6}$ and $\ln 2$, we can see that

$$(\ln 2)^3 \approx 0.333025$$

and

$$\ln 2 \frac{\pi^2}{6} = 1.140182$$

. We could certainly continue from here, and get our answers; however, if we wanted an exact answer, we would never get it.

5. Conclusions

Linear quantities and comparisons can be used on a wide variety of rational numbers, and have applications such as Diophantine equations and general algebra. However, irrational and transcendental numbers cannot be compared in this way (with a finite amount of steps), as having a finite number of steps would necessarily imply rationality. Despite this, we can sometimes approximate non-rational numbers to somewhat compare them in this fashion.