A Method for Finding Large Primes: Euler's Method

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1 Abstract

Prime numbers are a valuable concept in number theory which play an important role in the fields of mathematics and computer science. Prime numbers have fascinated mathematicians for the patterns they display. As a result, mathematicians have created methods for finding even the largest prime numbers. One such person is Leonard Euler, a Swiss polymath who made numerous influential discoveries in various branches of mathematics such as analytic number theory, complex analysis, and infinitesimal calculus. We present Euler's method for generating a significant number of large prime numbers, which involves a systematic approach in which mathematicians can meticulously categorize composites through the examination of factorization. Euler's method continues to hold historical significance till this day.

2 Introduction

Adrian Dudek, a mathematician from the Australian National University, once said: "I think the fascination for prime numbers comes from the fact that they are so elementary in description but yet incredibly difficult to analyse. A young child can understand what makes a number prime, yet lifetimes of mathematical research have been spent trying to solve some of the problems in the field."

Ever since the times of the ancient Greeks, mathematicians have been fascinated by the concept of prime numbers. Prime numbers, otherwise known as "primes", must satisfy the following conditions: is a natural number, greater than 1, and has only 2 divisors - 1 and itself. Consider the numbers 2, 3, 5, 7, 11, and 19, which are prime numbers because they have exactly two factors, one and itself.

On the other hand, numbers like 4, 6, 8, 9, 10 and 12 are not prime numbers because they have more than two factors. Specifically, non-prime numbers should satisfy the following conditions: not a natural number (e.g. a fraction, like 1/2 or 14/5, less than or equal to 1 (e.g. not prime: -3, 0, 1), can be formed by multiplying 2 natural numbers less than itself (e.g. not prime: 12 = 3 * 4).

Compared to number sequences such as the square numbers (i.e., 1, 4, 9, 16, 25...), whose rule is to simply add 2 more than you did before, the prime numbers do not display such a rule for their acquisition. Therefore, to answer a question such as "What prime number comes after 1,000,000?" requires a considerable amount of effort. Till this day, mathematicians continuously attempt to understand the distribution of prime numbers among the natural numbers - including at short and large scales.

One such Mathematician includes Leonard Euler (1707–1783). Euler was a a Swiss physicist, mathematician, geographer, astronomer, logician and engineer whose discoveries greatly influenced the field of mathematics. Perhaps one of greatest findings includes an easy method for finding "very many large prime numbers" [1].

3 Primer on Prime Numbers

We know that a prime number, or 'p', is specifically an integer that is greater than or equal to 2 and where the only two factors that it possesses are 1 and itself. We can also express 'p' as prime if it doesn't have any other proper factors side from 1 and 'p'. 'p' can only be found 25 times between 1 and 100. Here are the primes within that range:

'P' (1-10): 2, 3, 5, 7

'P' (11-20): 11, 13, 17, 19

'P' (21-30): 23, 29

'P' (31-40): 31, 37

'P' (41-50): 41, 43, 47

'P' (51-100): 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

Now, we need to start consider a very important concept: unique factorization. To do this, let's consider a number like 17,120,443. Such a number can be inherently broken down into its prime 'p' factors. This prompts a question: is this factorization unique, disregarding the order? And, could there be any numbers that can be factored into primes in more than just a single way.

We come to find that we can actually express the number, 17,120,443, in several different factorizations. Like such:

17,120,443 = 3599 × 4757

17,120,443 = 3953 × 4331

17,120,443 = 4087 × 4189

All of these factorizations are certainly distinct, but we should consider if any of these factors prime. Once we take a closer look, we can begin to see that none of the factors found in those expressions of the number 17,120,443 are prime within themselves. In fact, the prime factors of 17,120,443 are the following:

17,120,443 = 59 × 61 × 67 × 71

Now, let's test whether a certain number, say 'n', is a prime number 'p'. The "brute force approach" involves assessing all of the integers that are between 1 and the square root of 'n' to check if any of them can be factors of 'n'. There's no need to check potential factors that are larger than the square root of 'n'.

However, this method appears to be unmanageable when 'n' is a very large size. For instance, let's consider the largest known prime number mathematicians have every found: $2^(82, 589, 933)1$. This number is also known as a Mersenne number, or one less than a power of 2. When attempting to calculate, the sheer number of calculations are significant.

Due to this, mathematicians required a method to optimize the calculation process and find prime numbers on the far end of the number line. This led to the development of more efficient and rapid testing algorithms and specialized techniques for identifying candidate primes that have a higher likelihood of actually being prime numbers.

4 Euler's Method

Euler's method, as elucidated in his paper "An easy method for finding many very large prime numbers," presents a systematic approach to identifying a significant number of very large prime numbers using the formula $232aa+1$. The following breakdown elaborates on the mathematical intricacies of this method.

4.1 Composite Number Expressions

Euler begins by examining the composite number expressions that arise when the formula $232aa + 1$ yields a composite result. Euler begins by examining the composite number expressions that arise when the formula $232aa + 1$ yields a composite result. Here, a is a variable representing an integer, and Euler establishes the relationship $232aa + 1 = 232x + y$, exploiting the fact that $232 = 8 \times 29$ is a numerus idoneus. This leads to the equation $232(a-x) = y-1$, indicating that $a - x$ must be divisible by 232.

4.2 The Role of Variable 'z'

To further analyze the divisibility conditions, Euler introduces a variable 'z'. By substituting $yy - 1$ as $\frac{1}{2}z(29z \pm 1)$, he uncovers the underlying relationships between 'a' and 'x', which influence whether the formula yields composite or prime numbers. This pivotal step allows Euler to connect the divisibility properties with the values of 'z' and consequently the values of 'a'.

4.3 Identifying Exclusions

Euler systematically explores different values of 'z' to identify which 'a' values result in composite numbers. He notices that to yield a composite number, the expression $\frac{1}{2}z(29z \pm 1)$ must be factorizable into two distinct integers 'r' and 's'. This requirement leads to a systematic analysis of factorizations and exclusions for various values of 'z'.

4.4 Generating Exclusion Table

For each value of 'z', Euler resolves the factorization $\frac{1}{2}z(29z \pm 1) = rs$ to generate a list of excluded 'a' values. The exclusions stem from the requirement that both 'r' and 's' must be of the same parity, either both even or both odd. These exclusions correspond to the 'a' values for which the formula produces composite numbers.

Euler's systematic analysis culminates in a table that lists the excluded values of 'a' for each corresponding value of 'z'. This table succinctly captures the values of 'a' that lead to composite numbers when plugged into the formula $232aa + 1$. Below is the final table of exclusions as presented in Euler's paper:

Value of z	**Excluded a Values**
1	4, 8
2	11, 29, 30
3	14, 23, 34, 65
4	-
5	19, 21, 23, 33, 39, 47, 91, 183
6	23, 25, 41, 55, 88, 89, 260, 263
7	54
8	32, 40, 80, 232, 234
9	70, 198
10	51, 56, 147, 148, 244

Table 1: Table of Excluded Values of 'a' for Different Values of 'z'

This table serves as a key outcome of Euler's method, outlining precisely which 'a' values are to be excluded to ensure the formula generates prime numbers for the remaining 'a' values. Euler's analytical approach, meticulously documented in this table, showcases his mathematical ingenuity in systematically identifying exclusions and prime number candidates.

4.5 Determining Remaining Prime Candidates

Euler's genius lies in realizing that the 'a' values not listed in the exclusion table are prime number candidates. These values, when plugged into the formula $232aa + 1$, result in prime numbers. By systematically excluding composite cases, Euler unveils a method for generating a substantial number of very large prime numbers.

4.6 Modern Significance

Euler's method exemplifies algorithmic thinking applied to number theory. His systematic approach and the notion of exclusions laid the foundation for modern algorithms that efficiently generate prime numbers. Euler's method continues to influence contemporary mathematical research and computer-based techniques for prime number identification.

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4.7 Final Table of Exclusions

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Table 2: Table of Excluded Values of 'a' for Different Values of 'z'

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5 Conclusion

In conclusion, prime numbers stand as a timeless enigma in the realm of mathematics, captivating the minds of both budding enthusiasts and seasoned researchers. Their simplicity in definition, juxtaposed with their intricate distribution, has fueled generations of mathematicians' fascination and inquiry.

Leonard Euler, a luminary in the annals of mathematical history, presented an innovative method that unveiled a treasure trove of very large prime numbers. His method, as elucidated in his paper "An easy method for finding many very large prime numbers," exemplifies the prowess of algorithmic thinking applied to number theory.

Euler's systematic approach, showcased in his analysis of the formula $232aa+1$, introduced a framework for identifying prime numbers amidst the vast space of potential composites. By meticulously categorizing excluded values through a rigorous examination of factorizations, Euler demonstrated how exclusions can be the key to discovering prime number candidates. His groundbreaking work laid the groundwork for modern algorithms and computational techniques that continue to play a crucial role in prime number generation.

Beyond its historical significance, Euler's method holds contemporary relevance. It underscores the enduring influence of mathematical insights on modern computational endeavors. The journey from the abstract realm of number theory to practical applications in computer science is beautifully exemplified by Euler's contributions.

In conclusion, Euler's method shines as a testament to the symbiotic relationship between theoretical mathematics and applied computation. As we delve deeper into the intricate patterns woven by prime numbers, Euler's legacy serves as a guiding light, inspiring mathematicians and computer scientists to unravel the mysteries of the mathematical universe.

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7 Citations

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