Various Geometric Demonstrations

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ABSTRACT

In this paper, I will discuss the various geometric demonstrations that Euler was referring to. In his paper, he proves several small theorems in Euclidean geometry, and I will go through them and reprove them with my own words and illustrations.

The paper commences with a proof of a basic lemma about a line that is split at 2 arbitrary points. This is Lemma 1 which is proved with some basic arithmetic. This Lemma is brilliantly applied in the next proof.

The next proof that is covered in this paper is that of one of Fermat's theorems. It involves the usage of arithmetic, geometric logic, and Lemma 1. The theorem has to do with a relationship between a semicircle, a cord, and a parallelogram.

The next proof that is covered is that of a circle inscribed in a triangle. We show that the sum of the sides, multiplied by half of the radius of the inscribed circle, is the total area of the triangle. We prove this with geometric manipulation.

Another theorem follows. This one follows with the concept of the semiperimeter. We are now considering different ways to write the semiperimeter in terms of side lengths. This proof will help us continue the paper and come to new conclusions.

The final proof that is covered in this paper is that of Heron's formula. Heron's formula is a commonly used formula that can determine the area of any triangles with just the side lengths. This paper uses mathematics, but it also creates a very clear visual as to why this formula works. Another brilliancy by Euler.

To close this paper off, I have some concluding remarks.

The first proof that Euler covers is a simple one involving a line segment ABCD. Euler proposed the following lemma.

Lemma 1: If a straight line AD is arbitrarily at 2 points, B and C, then the rectangle formed with side lengths BC and AD combined with the rectangle formed with side lengths AB and CD would have equivalent area to the rectangle formed with side lengths AC and BD.

$$AD * BC + AB * CD = AC * BD$$

We can prove this with some elementary algebra. Let us establish that AB+BC+CD=AD. Substitute AB+BC+CD for AD. We can rewrite it as follows.

$$AD * BC + AB * CD = AC * BD$$

= (AB+BC+CD)*BC + AB*CD

= BC(AB+CD)+(AB*CD)+BC^2 $= AB * BC + BC * CD + AB * CD + BC^2$

Now let us visit the left side of the original equation. AC=AB+BC and BD=BC+CD.

$$AD * BC + AB * CD = AC * BD$$

= (AB+BC)(BC+CD)

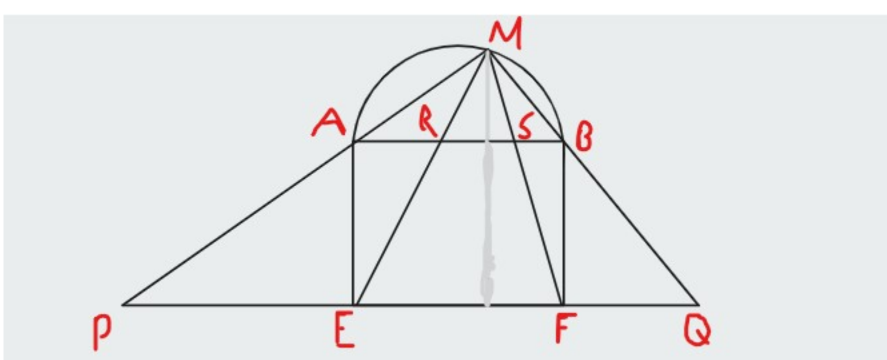
= AB*BC+BC*CD+BC^2 $+ AB * CD$

Now we are left with the following:

$$AB * BC + BC * CD + AB * CD + BC^2 = AB * BC + BC * CD + BC^2 + AB * CD$$

Which is clearly 2 equivalent statements.

Theorem 1 (Fermat's Theorem): If upon the diameter of AB of the semicircle AMB, there is a parallelogram ABFE, whose latitude AE or BF is equal to the chord of a quarter of the same circle or to the side of the inscribed square, and from points E and F there are two lines EM and FM down to any point M on the periphery, then the diameter AB will be cut at the points R and S, such that: $AS^2 + BR^2 = AB^2$



Let us start of by extending MA to MAP and MB to MBQ so that PQ contains EF. $\angle AMB$ is a right angle, $\angle P$ and $\angle Q$ add to be 90 degrees, as the angles of triangle PMQ must add to be 180 degrees. $\angle PEA$ and $\angle BFQ$ are also right angles, as AE abd BF are perpendicular to PQ, so $\angle P + \angle PAE = \angle Q + \angle FBQ = 90$ degrees.

$$\angle P + \angle Q = 90 = \angle P + \angle PAE \implies \angle Q = \angle PAE \implies \angle P = \angle FBQ \implies \text{PAE and BFQ are similar.}$$

Using the definition of similarity, $\frac{PE}{AE} = \frac{BF}{QF} \implies PE * QF = AE * BF$. We know that AE and BF are equivalent, so we can say $= AE^2 \implies 2PE * QF = 2AE^2$. AE is equivalent to the cord of the square inscribed in the circle, we have $2AE^2 = AB^2 = EF^2 \implies 2PE * QF = EF^2$.

Due to similarity, $2PE * QF = EF^2 \implies 2AR * SB = RS^2$. Now we can do some algebraic steps.

$$(AS + AR)^2 = (AB + RS)^2$$
$$\text{AS}^2 + BR^2 + 2AS * BR = AB^2 + RS^2 + 2AB * RS$$

Let us substitute $2 * AR * BRforRS^2$

$$AS^2 + BR^2 + 2AS * BR = AB^2 + 2AB * RS + 2AR * BS$$

Now here is the interesting part. Allow us to apply Lemma 1 to make it much simpler.

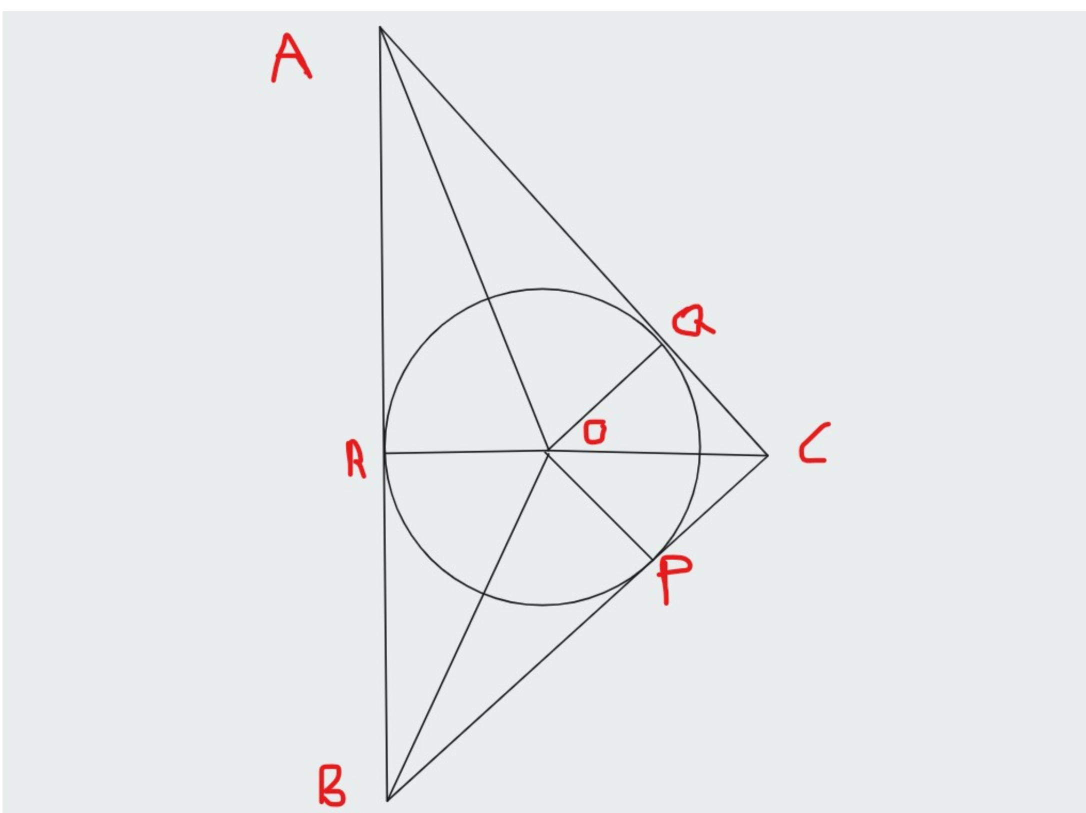
$$AB * RS + AR * BS = AS * BR \implies 2AB * RS + 2AR * BS = 2AS * BR$$

Now let us substitute this back into our original substitution.

$$AS + BR^2 + 2AS * BR = AB^2 + 2AS * BR \implies AS^2 + BR^2 = AB^2$$

Q.E.D.

Theorem 2: The area of any triangle ABC is equal to the product of half of the sum of the sides and the radius of the inscribed circle.



We are trying to prove the following:

$$ABC = \frac{1}{2}(AB + AC + BC)OP$$

Lines OP, OQ, and OR are each perpendicular to a side of the triangle. We also draw lines OA, OC, and OB. This will allow us to divide our triangle into AOB, AOC, and BOC. Now here comes the interesting part. If we line these triangles up, side by side, we would have 3 triangles, each with altitude OP, and combined base of AB+BC+AC. The area of these three triangles is therefore

$$\frac{1}{2}(AB + BC + AC)OP$$

and this is equal to the original triangle.

Q.E.D

Theorem 3: If we draw perpendicular lines OP, OQ, and OR from the center O of the circle inscribed in ABC, the sides will be cut by these lines in such a way that the posited semiperimeter S will be

AR = AQ = S - BC

BR = BP = S-AC

CP = CQ = S - AB

where S is half the sum of the side lengths.

Allow us to observe using the image from the question before. We know that the perpendicular lines, OP, OQ, and OR, are all radii of the same circle, and therefore equal. This leads us to conclude:

AQ = AR, BP = BR, and CP = CQ

and we can continue on this line of thought:

AB + BC + AC = AR + BR + BP + CP + CQ + AQ = 2AR + 2BP + 2CQ

Because we had $\frac{1}{2}(AB + BC + AC) = S$, we also have $AB + BC + AC = 2S$.

$$2S = 2AR + 2BP + 2CQ \implies AR + BP + CQ = S$$

And now, using our previous statements and alegbra, we can conclude the following:

$$AR + BC = S \implies AR = AQ = S - BC$$
$$BP + AC = S \implies BP = BR = S - AC$$
$$CQ + AB = S \implies CQ = CP = S - AB$$

Q.E.D.

Theorem 4: The area of any triangle ABC can be found, if the sides are subtracted separately from the semiperimeter (which is S) and the product of these three remaining sides is multiplied by the semiperimeter itself, and the square root of the product is taken.

What we are are trying to prove is the following:

$$\text{A}Area = \sqrt{S(S - AB)(S - AC)(S - BC)}$$

Before we begin this proof, I would like to state that this formula for the area of a triangle is commonly known as Heron's formula.

Let us start of by bringing up Theorem 2. We know that the area of a triangle can be expressed as S(OP) where S is the sempiperimeter. We also know that $S \cdot OP^2 = AR \cdot BP \cdot CQ$. Multiply both sides by S:

$$S^2 \cdot OP^2 = S \cdot AR \cdot BP \cdot CQ$$

Now we can take the square root of both sides

$$S \cdot OP = \sqrt{S \cdot AR \cdot BP \cdot CQ}$$

Since $S \cdot OP$ is the area, and so is $\sqrt{S \cdot AR \cdot BP \cdot CQ}$.

But what did we conclude in our last paragraph?

$$AS = S - BC, \ BP = S - AC, \text{ and } CQ = S - AB$$

Post substitution, we get

$$\text{Area of triangle ABC} = \sqrt{S(S - AB)(S - AC)(S - BC)}$$

Q.E.D.

Concluding Remarks

This paper covers several geometric proofs, or as Euler called it, "Variae Demostrationes Geometriae," translating to "Various Geometric Demonstrations". The only relationship between the theorems is that they are both in Euclidean geometry, whereas the theorems apply their respective Lemmas. Writing this was a good way to conclude my time at the Euler Circle.

I would love to continue my journey in mathematics and the Euler Circle has solidified that. I will continue to read the works of Euler, and I would love to create a project similar to this. I could not cover the entirety of this paper of Euler, but I covered the first half, and made it comprehensive in a manner that everything wove together nicely.