THE MANDELBROT SET

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The Mandelbrot set is a very famous set, named after mathematician and computer scientist Bernoit Mandelbrot. Its popularity is in part due to its aesthetic appeal once graphed upon the complex plane. Simply, it can be defined as the set M such that for $c \in M$, the iterates $f(0), f(f(0)), f(f(f(0))), \ldots$ where $f(z) = z^2 + c$. This paper is largely based on the work of [Dev89a].

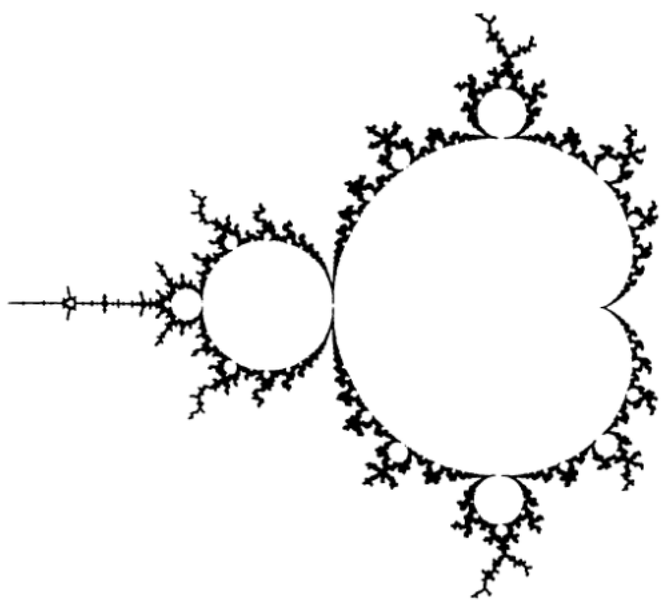

Figure 1. The points within the Mandelbrot Set plotted on the Complex Plane

1. Definitions

Definition 1. Let $F : \mathbb{C} \to \mathbb{C}$. The point x_0 is a fixed point for F if $F(x_0) = x_0$. The point x_0 is a periodic point of period n for F if $F^n(x_0) = x_0$ but $F^i(x_0) \neq x_0$ for $0 < i < n$. The point x_0 is eventually periodic if $F^n(x_0) = F^{n+m}(x_0)$, but x_0 itself is not periodic.

Definition 2. For a polynomial P the sequence

$$z_0, z_1 = P(z_0), \ldots, z_{n+1} = P(z_n), \ldots$$

is called the orbit of z_0 under iteration.

Notation. The concatenation of $P(P(\ldots(x))\ldots)$ k times is written as $P^{\circ k}(x)$, e.g. $P(P(3)) = P^{\circ 2}(3)$.

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Definition 3. For a periodic point z_0 of period k, the multiplier ρ is defined as the derivative of $P^{\circ k}$ at z_0. Via the chain rule, this gives

$$\rho = (P^{\circ k})'(z_0) = \prod_{j=0}^{k-1} P'(z_j),$$

so that the derivative of $P^{\circ k}$ is the same at all points of the cycle.

2. Periodicity

We call a cycle *attracting* if $|\rho| < 1$, *superattracting* if $\rho = 0$, *repelling* if $|\rho| > 1$, and *neutral* if $|p| = 1$. Note that a cyce is superattracting if and only if a critical point belongs to the cycle. The behavior of each type of cycle can be observed with the Taylor expansion of $P^{\circ k}$ about z_0:

$$P^{\circ k}(z_0 + u) = z_0 + \rho u + \dots.$$

Evidently, if $|\rho| < 1$ and z is sufficiently close to z_0 then $P^{\circ k}$ is closer to z_0 than z, $P^{\circ 2k}$ is even closer, and this process continues. Therefore an attracting cycle pulls points in near a relative basin to z_0. Similarly, if $|\rho| > 1$ and z is sufficiently close to z_0 then the iterate $P^{\circ k}$ is further away from z_0 than z. It is aptly named a repelling cycle. We can now begin a closer discussion of the polynomial $P_c(z) = z^2 + c$, which is, of course, the quadratic we are mostly focused on for the Mandelbrot set. A *critical point* ω is defined such that $P'(\omega) = 0$, or that $\omega = 0$ in this instance. The parameter value c is known as the *critical value*, or $P_c(0)$. The fixed points of P_c, given by the roots of $z^2 + c = z$ are

$$z = (1 \pm \sqrt{1 - 4c})/2$$

with multiplier

$$\rho = 1 \pm \sqrt{1 - 4c}.$$

These calculations can be easily verified. The periodic points of period 2 are the solutions to the equation $P_c P_c(z) = z$, or $(z^2 + c)^2 + c = z$. The roots of this equation which are not fixed points are

$$z = (-1 \pm \sqrt{-3 - 4c}).$$

The corresponding multiplier is given simply by $\rho = 4(1 + c)$. A relevant theorem whose proof can be found in [Bro65] is the following:

Theorem 4 (P. Fatou)**.** *Every attracting cycle for a polynomial attracts at least one critical point.*

A polynomial of degree $d \geq 2$ can therefore have at most $d - 1$ attracting cycles in the plane. The polynomial P_c therefore can have at most one attracting cycle. This fact will be used later in this paper.

3. The main cardioid

The main cardioid of the Mandelbrot Set is the most visually apparent feature of the fractal, and in fact it turns out to be very easy to see why it occurs. We turn our focus strictly to the fixed points of P_c, i.e. those with multiplier

$$\rho = 1 \pm \sqrt{1 - 4c}.$$

We arrange to solve for c in the abbreviated steps:

$$\rho = 1 \pm \sqrt{1-4c}$$
$$\rho - 1 = \pm\sqrt{1-4c}$$
$$(\rho-1)^2 = 1-4c$$
$$-\rho^2 + 2\rho = 4c$$
$$c = \rho/2 - (\rho/2)^2.$$

The unit circle $\rho = e^{2\pi it}$ for $t \in \mathbb{R}/\mathbb{Z}$ in the ρ-plane correspond to

$$e^{2\pi it}/2 - (e^{2\pi it}/2)^2$$

in the c-plane. This parameterized figure turns out to indeed be the big cardioid seen in Figure 1. The subset W_0 is the subset of M bounded by this cardioid. W_0 is an example of a hyperbolic component, a topic that will be further analyzed in the next section. An equivalent definition of W_0 is that $W_0 = \{c \in \mathbb{C} | P_c \text{ has an attracting fixed point}\}$.

4. Hyperbolic components

Theorem 1 states that any c-value for which there exists an attracting cycle is contained in M. Let $H(M) = \{c \in \mathbb{C} | P_c \text{ has an attracting cycle}\}$. Recall that the attracting cycle is unique. A connected component W of $H(M)$ is called a *hyperbolic component* of M. A conjecture that has been dubbed possibly "the most important open problem in the field of complex dynamics" the following:

Conjecture 5 (The hyperbolicity conjecture for polynomials in degree 2)**.** *The set of c-values for which P_c is hyperbolic is $\mathbb{C}$. Equivalently that $H(M)$ equals the interior of M: int$(M) = H(M)$.*

Under our definition of a hyperbolic component, it is clear that the main cardioid also must be a hyperbolic component. We begin finding more by considering the subsets of M for which P_c has an attracting fixed point or an attracting cycle of period 2. We let $W_{\frac{1}{2}} = \{c \in \mathbb{C} | P_c \text{ has an attracting cycle of period 2}\}$. Similar to the situation with the main cardioid, it turns out this is not so hard to find out. Recalling that the multiplier for the periodic points of period 2 is $\rho = 4(1+c)$, we solve for c to get $c = \rho/4 - 1$. We then replace ρ for the unit circle in the ρ-plane, yielding $c = \frac{1}{4}e^{2\pi it} - 1$. This immediately tells us that $W_{\frac{1}{2}}$ is an open disc centered at $c = -1$ with radius $\frac{1}{4}$. Referring back to Figure 1, we can verify that there is, in fact, a disk just to the left of the main cardioid. The closure of W_0 and $W_{\frac{1}{2}}$ share the point $\{-3/4\}$ in common. $\{-3/4\}$ is known here as the *period doubling bifurcation* point, or where c changes from the disc to the main cardioid. It is shown in [Yoc86] that for each $\frac{p}{q}$ where p and q are coprime that there is a hyperbolic component $W_{\frac{p}{q}}$ satisfying

$$\overline{W}_{\frac{p}{q}} \cap \overline{W_0} = c$$

where c is a *period q-doubling bifurcation* point. This is when c changes from being inside the big cardioid to inside the hyperbolic component $W_{\frac{p}{q}}$, then undergoes a cycle of period q changes from being repelling to attracting after some a_c changes from being attracting to repelling. The set $M^*_{\frac{p}{q}}$ = the connected component of $M - \overline{W_0}$ containing $W_{\frac{p}{q}}$. The limb $M_{\frac{p}{q}}$ is the intersection of $M^*_{\frac{p}{q}}$ and c.

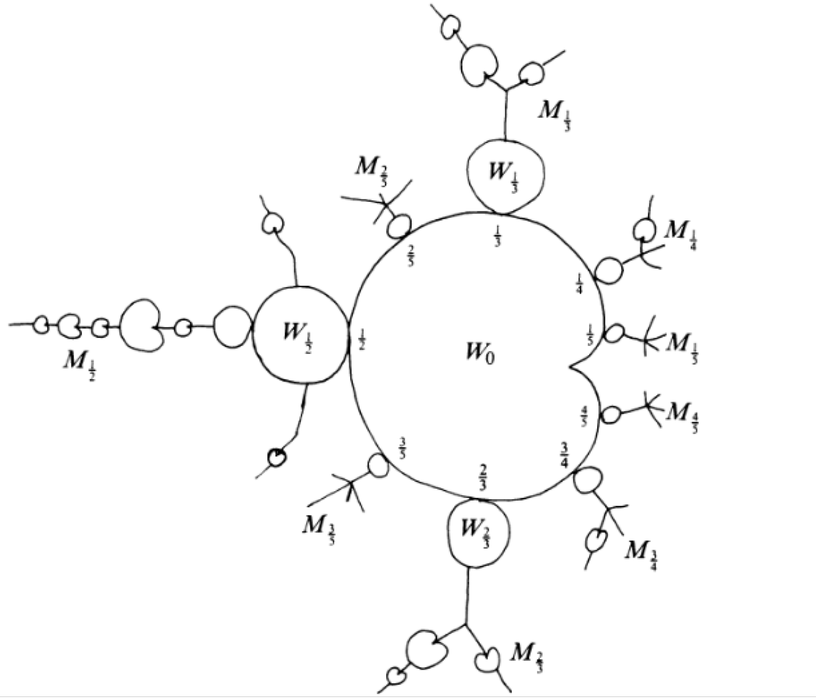

Figure 2. The Mandelbrot Set with labeled hyperbolic components

5. Chaos Theory

This paper ends with a short discussion on chaos theory, and a family of quadratics highly related to the Mandelbrot Set. Consider now the function $F_c(x) = x^2 + c$. When $c > \frac{1}{4}$ this is relatively uninteresting as there are no fixed points. For $c = \frac{1}{4}$, f has a fixed point at $x = \frac{1}{2}$. An interesting phenomena occurs as c tends to decrease. Labeling the fixed points p_+ and p_- for the larger and smaller ones, respectively, for $-3/4 < c < 1/4$ and $p_- < x < p_+$ we determine $F_c^{\circ n} \to p_-$. However if $|x| > p_+$, then $F_c^{\circ n}(x) \to \infty$. This means that one fixed point has split into two; this is a bifurcation. If $|x| > p$ and $c < \frac{1}{4}$, $F_c^{\circ n}(x) \to \infty$ as well. Therefore we focus all of our attention to the range $-p_+ < x < p_+$.

Our next bifurcation occurs when $c = -\frac{3}{4}$. This is evident due to the function $F_c^{\circ 2}(x) = (x^2 + c)^2 + c$.

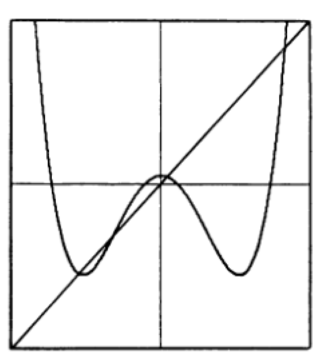

Figure 3. $F_c^{\circ 2}(x)$ plotted alongside $y = x$ for $c < -3/4$

For each minima on F_c^2, note that the graph "resembles" the graph of F_c. It is here that we expect that F_c^2 will behave dynamically similar to F_c did in a larger interval. Plotting F_c^3, we see that this phenomena is exactly true. Essentially, we expect that F_c^2 will attain derivative -1 and undergo its own period doubling transformation. In this way, our period 2 orbit will turn into a period 4 orbit. In fact, there is nothing stopping this period 4 orbit doubling into a period 8 orbit, and that into a period 16 orbit itself. Continuing, we expect a sequence of c-values $c_1, c_2, \dots c_n \dots$ at which F_c undergoes a period doubling transformation to have a period 2^n. This indeed happens, but a rigorous proof is deflected to [Fei78].

This sequence of bifurcations eventually seems to disappear for decreasing values of c, and the period of the orbit begins to vary erratically. This sequence is called *the period doubling route to chaos*. Further analysis can and has been done on this topic, and the reader is advised to pursue it in [Dev89b]. The truly chaotic nature of this sequence, however, is truly observed by plotting the orbits of F_c for decreasing values of c. The figure below is the last 100 of 150 iterates of 0 for 300 equally spaced c-values between $\frac{1}{4}$ and -2. The reader is encouraged to observe the descent from an orderly doubling of period into the abyssal bands of chaos on the right of the graph.

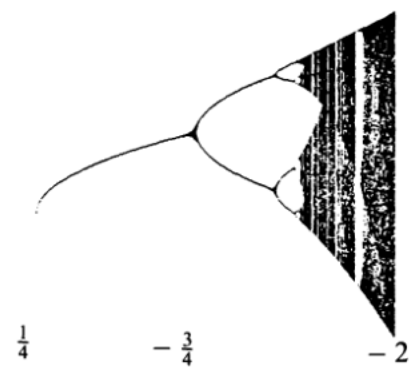

Figure 4. The bifurcation diagram of F_c

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