ERGODIC THEORY WEEK 10: ERGODIC THEORY OF NON-INTEGER BASES

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1. An Introduction to Base-β Expansions and the T_β transformation[5]

For any integer $\beta > 1$, we can express every $x \in [0, 1)$ as

$$x = \sum_{n=1}^{\infty} \frac{d_n}{\beta^n}$$

where $0 \leq d_n < \beta$ and $d_n \in \mathbb{Z}$ for all natural numbers n. We call this expression the base-β expansion of x. These expansions are also referred to as *greedy expansions*, because they are created using the greedy algorithm - we take as many of the largest power of β as we can before adding to digits of lower place-values.

Example. Let us express the number 7_{10} in base 3, ternary. The largest power of 3 less than or equal to 7 is $3^1 = 3$. 3 goes into 7 two times maximum, leaving a remainder of 1. So, we write out the base 3 expansion in the following way. The place-values are defined by powers of the base. Like in base 10, how we have the 10's place to the left of the ones place, the right-most place before the 3-imal point is the ones place, and the place to the left of that is the 3^1's place. So, since $7 = 3^1 \times 2 + 1$, we write 7 in base 3 as 21_3.

In fact, any number can be expressed in any base β, even when β is not an integer, as long as $\beta > 1$. It is easy to see that the digits base β would be $(0, 1, ...\lfloor\beta\rfloor)$. Note that this means it is impossible to express a number in base β when $\beta \leq 1$, because the only digit allowed would be 0.

Example. Let us consider, for example, the number 3 in base ϕ, the golden ratio or golden mean, which is equal to $\frac{1}{2}(\sqrt{5}+1)$. The powers of ϕ can be approximated in base 10 as $\phi = 1.618..., \phi^2 = 2.618..., \phi^3 = 4.236....$ Clearly, the largest power of ϕ less than or equal to 3 is $\phi^2 = 2.618...$, which, when subtracted from 3, gives us a remainder of $3-\phi^2$. The largest power of ϕ which goes into $3 - \phi^2$ is $\frac{1}{\phi^2}$, which goes in perfectly, giving us and expansion of

$$3_{10} = 100.01_\phi$$

If we create non-integer expansions without following the greedy algorithm, we do not always have unique representation. For example,

$$0.1_\beta = 0.011_\beta$$

where β is the golden ratio. Thus, we always use the greedy expansion when creating a β-expansion for any number α.

Recall (from Week 4 Problems 6 and 7) the transformation $T_\beta : [0, 1) \to [0, 1)$ defined by

$$T_\beta(x) = \beta x - \lfloor \beta x \rfloor = \beta x \pmod 1.$$

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Figure 1. The Transformation T_5

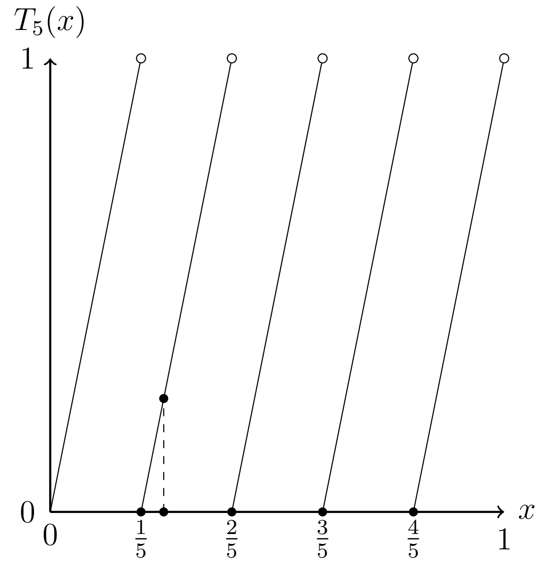
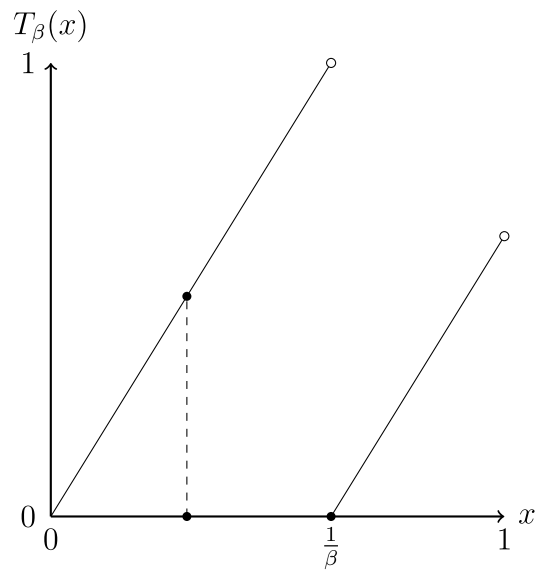


Figure 2. The Transformation T_β where β is the golden ratio



This transformation is clearly measure preserving when β is an integer. However, we're interested in when β is not an integer.

2. Invariant Measures for T_β

Theorem 2.1. *If $\beta > 1$ is not an integer, T_β is not measure-preserving with respect to the Lebesgue measure, λ.*

Note that it's fairly easy to see this from Figure 2.

Proof. If $a, b \in [0, 1)$, such that $\beta - \lfloor\beta\rfloor < a < b < 1$, then

$$T_\beta^{-1}((a,b)) = \bigcup_{j=0}^{\lfloor\beta\rfloor-1} (\frac{a}{\beta} + \frac{j}{\beta}, \frac{b}{\beta} + \frac{j}{\beta})$$

because $T_\beta(x)$ depends only on where it is placed between multiples of $\frac{1}{\beta}$, and therefore numbers that are $\frac{n}{\beta}$ apart for $n \in \mathbb{Z}$, $n < \beta$ will map to the same number when the transformation T_β is applied. We can find the Lebesgue measure of this to be

$$\lambda(T_\beta^{-1}(a,b)) = \sum_{j=0}^{\lfloor\beta\rfloor-1} \frac{b-a}{\beta} = \frac{\lfloor\beta\rfloor}{\beta}(b-a).$$

Since β is not an integer, $\frac{\lfloor\beta\rfloor}{\beta}$ will be less than one, so $\lambda \circ T_\beta^{-1}((a,b)) < \lambda((a,b))$ ∎

Even so, there is still something interesting that we can do with T_β and the Lebesgue measure.

Theorem 2.2. *All T_β-invariant sets have measure zero with respect to the Lebesgue measure for all $\beta > 1$. [2]*

Proof. First, let B be a T_β-invariant set with positive Lebesgue measure, and let $\mathcal{C}$ be the collection of all fundamental intervals. If $E \in \mathcal{C}$, we have

$$\frac{\lambda(B \cap E)}{\lambda(E)} = \frac{\lambda(T^{-n}(B) \cap E)}{\lambda(E)} = \frac{\lambda(B \cap T^n(E))}{\lambda(T^n(E))} = \lambda(B).$$

We can now apply Knopp's lemma with $\gamma = \lambda(B)$, so $\lambda(B) = 1$. ∎

However, it has been proven that there exists an invariant measure ν_β for all T_β where $\beta > 1$. Furthermore, it has been proven that this ν_β is *equivalent* to the Lebesgue measure.

Definition 2.3. A measure μ is said to be *equivalent* to the Lebesgue measure if μ and λ have the same sets of measure zero.

Note that because λ and ν are equivalent, T_β must be ergodic with respect to λ.

Theorem 2.4. *There exists a measure ν of the form $\nu_\beta(A) = \int_A h_\beta(x)\,dx$, where h_β satisfies $0 < h_\beta(x) < \infty$, and $\nu_\beta(T_\beta^{-1}(A)) = \nu_\beta(A)$ for all $\beta > 1$.*

This theorem is very difficult. A proof can be found at [4]. In fact, this invariant measure has been found explicitly, in another very difficult paper [3] to be

$$\nu_\beta(A) = \int_A h_\beta(x)\,dx$$

where

$$h_\beta(x) = \frac{1}{F(\beta)} \sum_{x < T_\beta^n(1)} \frac{1}{\beta^n}$$

for $x \in [0, 1)$, where the sum is over all nonnegative n such that $x < T_\beta^n(1)$, and the normalizing constant is

$$F(\beta) = \int_0^1 \sum_{x < T_\beta^n(1)} \frac{1}{\beta^n}\,dx$$

3. Analogue of Normality

Definition 3.1. Recall from week 4 that if $\alpha \in [0,1)$, we say that α is *simply normal* in base $b > 1$, where b is an integer if we have

$$\lim_{N\to\infty} \frac{\#\{i : 1 \le i \le N \text{ and } d_i = d\}}{N} = \frac{1}{b},$$

where $\alpha = 0.d_1 d_2 ...$, for all integers $0 \le d < b$. In other words, the digits of α are uniformly distributed.

The analogue of normality for base β is that $(x\beta^n)_{n\in\mathbb{N}}$ is uniformly distributed mod 1. [1] More generally,

Definition 3.2. If $\alpha = 0.d_1 d_2 \ldots$ is the β-expansion of $\alpha \in [0,1)$, we say that α is *simply normal* in base β, where $\beta > 1$ if we have

$$\lim_{N\to\infty} \frac{\#\{i : 1 \le i \le N \text{ and } d_i = d\}}{N} = \frac{1}{\lfloor\beta\rfloor},$$

for all integers $0 \le d < \lfloor b \rfloor$.

In week 6, we proved that almost all numbers are simply normal in any integer base b, but unfortunately this is not the case for non-integer bases.

Example. Consider the base β, where $\beta = \frac{1+\sqrt{5}}{2}$ is the golden ratio. We can calculate that $F(\beta) = \frac{1}{2}(5-\sqrt{5})$, which yields

$$h_\beta(x) = \begin{cases} \frac{5+3\sqrt{5}}{10} & 0 \le x < \frac{\sqrt{5}-1}{2} \\ \frac{5+\sqrt{5}}{10} & \frac{\sqrt{5}-1}{2} \le x < 1 \end{cases}$$

Because the resulting measure ν_β is equivalent to the Lebesgue measure, we know that T_β is ergodic with respect to ν. We can now apply the Birkhoff Ergodic Theorem to calculate the frequency of a given block of digits. For almost all $x \in [0,1)$, we have

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n}\#\{1 \le i \le n : d_i(x) = 0\} \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_{[0,\frac{1}{\beta})} \circ T_\beta^i(x) \\ &= \nu_\beta([0, \frac{1}{\beta})) \\ &= \int_0^{\frac{1}{\beta}} \frac{5+3\sqrt{5}}{10}\, dx = \frac{5+\sqrt{5}}{10} \approx 0.7236\ldots, \end{aligned}$$

so a.e. $x \in [0,1)$ contains $\approx 72.36\%$ zeroes in its base-β expansion, where β is the golden ratio.

References

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[3] Alexander Osipovich Gel'fond. "A common property of number systems". In: *Izvestiya Rossiiskoi Akademii Nauk. Seriya Matematicheskaya* 23.6 (1959), pp. 809–814.

[4] Alfréd Rényi. "Representations for real numbers and their ergodic properties". In: *Acta Mathematica Hungarica* 8.3-4 (1957), pp. 477–493.

[5] Nikita Sidorov. "Expansions in Non-Integer Bases". In: *School of Mathematics, The University of Manchester* (2010). URL: `http://www.maths.manchester.ac.uk/~nikita/qmul-2010.pdf`.